A strong completeness theorem for CTL

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Starting from A. Prior [3] modal logics of discrete time are used in the philosophical investigations of time and determinism. Logics LTL and CTL are modal logics of infinite discrete linear and branching time, respectively. They are also widely used in the model theoretic verification of programs where the finite state program is presented as a Kripke model and the properties of all possible computations are expressed by formulas of LTL or CTL. The logic LTL is interpreted in the frames of ω-type ordering while the logic CTL is interpreted in the frames being infinite trees with no finite branches.

The set of tautologies of LTL were axiomatized in [2]. An axiomatization of tautologies of CTL was given in [1]. When axiomatizing the set of tautologies of the above logics there is no need to use infinite rules of inferences. However, when one is interested in systems with the strong completeness property, the infinitary systems are needed. This is so, because logics in questions are not compact. Indeed, the set of formulas $\{p, Xp, X^2p, \ldots\} \cup \{\neg Gp\}$ is not satisfiable while any of its finite subsets is satisfiable.

The system is strongly complete for a logic L if it is complete with respect to any theory T in L. Then, the deductive system with strong completeness property for LTL was given by [5], see also [4].

It has been an open problem to construct an axiomatic system for CTL with strong completeness property. We present such a system strongly complete within a class of infinite finitely branching trees. Like Sundholm, we use a proper finitization of rules resulting with formulas of the form QG and QF, where $\mathrm{Q} \in \{\forall, \exists\}$. The restriction of semantics to finitely branching trees, due to Weak König Lemma, ensures that the rules for $\exists$ G are sound. Moreover, we add continuum many rules to ensure the inconsistency of theories which forces infinite branching of their models. To finitize these type of rules we consider two types of inferences: $\vdash_f$ and $\vdash_\omega$, where the former is a finitary version of the latter. According to $\vdash_f$, if a theory forces at a tree level n more than $f(n)$ nodes, it is inconsistent.

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[3] A. N. Prior, *Time and Modality*, Oxford University Press, Oxford, 1957.

[4] K. Segerberg, *Von Wright's Tense Logic*, In: P. A. Schilpp and L. E. Hahn (eds.), The Philosophy of Georg Henrik von Wright, Library of Living Philosophers, Vol. 19, Open Court, 1989, pp. 603–635. Written in 1974.

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