Kazuyuki Tanaka's work in reverse mathematics and second-order arithmetic

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This talk presents an overview of Professor Kazuyuki Tanaka's profound and enduring contributions to mathematical logic, with a particular emphasis on his foundational work in reverse mathematics and subsystems of second-order arithmetic. In honoring his memory, we will examine several central themes of his research, illustrating how his insights elegantly bridged model theory and proof theory.

One of Tanaka's most prominent research areas is the analysis of determinacy for finite Borel hierarchies [5, 6]. In particular, he studied determinacy axioms for the difference hierarchies of Σ^0_1 and Σ^0_2. From these axioms, he extracted new set existence principles that extend beyond the traditional "Big Five" subsystems. Together with many of his students, he clarified the complex structure of these determinacy systems [2]. This structural analysis was truly his lifework.

Whenever Tanaka discussed the highlights of his research, he always mentioned the conservation theorem that S. Simpson referred to as "Tanaka's Conjecture" [4]. This theorem generalizes Harrington's classical result by showing that WKL_0 is conservative over RCA_0 not only for Π^1_1 sentences, but also for sentences asserting unique existence ($\exists$!). It carries many intriguing implications; for instance, it can be used to derive a proof of the Fundamental Theorem of Algebra within RCA_0.

Tanaka's Self-Embedding Theorem for $\mathbf{WKL}_0$

The core of this talk focuses on Tanaka's celebrated 1997 self-embedding theorem [8], a key result in the model theory of second-order arithmetic. The theorem states that every countable nonstandard model (M, χ) of WKL_0 has a proper initial segment $\mathrm{I} \subseteq_e \mathrm{M}$ such that the cut I, equipped with the coded sets of M bounded in I, is isomorphic to the original model (M, χ). This theorem generalizes Friedman's classical self-embedding theorem for Peano arithmetic [11] to second-order arithmetic. Through subsequent analyses by researchers such as Wong and Enayat [1, 10], this result has led to important characterizations of many models in both first-order and second-order arithmetic. Furthermore, a major application of this theorem is that it allows nonstandard analysis to be applied effectively within reverse mathematics [7]. By using this framework, we can formalize nonstandard arguments and make complex proofs much more straightforward, such as the construction of the Haar measure [9] and the proof of the Jordan Curve Theorem [3].

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