Reals as naive Cauchy sequences in reverse mathematics

Patrick Uftring

Representing reals as fast-converging Cauchy sequences has a long history and comes with several advantages: Equality is Π^0_1 and, e.g. in the field of reverse mathematics, finite sums and maxima can be defined without the use of Σ^0_2-collection. Moreover, certain values such as bounds (in form of natural numbers) can be computed and signs of non-zero inputs can easily be searched for. For these and many more reasons, fast-converging Cauchy sequences are established as the preferred way to talk about reals, not only in the context of reverse mathematics (cf. [3, Sections I.4 and I.8]) but also in proof mining as well as computable analysis.

There are two issues with this approach: First, fast-converging Cauchy sequences are an ad-hoc solution to the raised problems. They do not, however, appear in first year textbooks on analysis. Hence, an argument can be made that they do not accurately capture what reverse mathematics intends to analyze. Secondly, statements such as the Bolzano-Weierstraß theorem quickly reach the strength of arithmetical comprehension. Here, we may ask if it is possible to develop basic analysis *strictly* below this system.

In this talk, we present joint work with Anton Freund and Nicholas Pischke (cf. [1]), in which we explored a substantial amount of real analysis using reals in the form of Cauchy sequences without any restrictions on their modulus. While some statements such as the mentioned existence of finite sums or maxima slightly increase in strength, others such as the Bolzano-Weierstraß theorem now stay strictly below arithmetical comprehension (this particular example has already been discussed by Kreuzer, [2]). Moreover, the usual path to arithmetical comprehension via weak Kőnig's lemma is surprisingly shifted to weak consequences of Ramsey's theorem such as Σ^0_2-collection or the cohesiveness principle.

[1] A. Freund, N. Pischke, and P. Uftring, *Avoiding logical strength in real analysis*, submitted, arXiv:2605.15151.

[2] A. P. Kreuzer, *The cohesive principle and the Bolzano-Weierstraß principle*, Mathematical Logic Quarterly, vol. 57 (2011), n^o 3, pp. 292–298.

[3] S. Simpson, *Subsystems of Second Order Arithmetic*, Perspectives in Logic, Cambridge University Press, 2009.