

On a gap in Scott spectra of

$I\Sigma_n + exp + \neg B\Sigma_n$

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By a classical result of Scott, for every countable structure M there is a sentence of $\mathcal{L}_{\omega_1, \omega}$ — an extension of first-order logic with countable conjunctions and disjunctions — in the signature of M , which is satisfied by a countable structure N if and only if N and M happen to be isomorphic. We call such a sentence a Scott sentence of M . Furthermore, formulas in $\mathcal{L}_{\omega_1, \omega}$ admit a hierarchy similar to the arithmetical one, starting with quantifier-free formulas denoted as $\Sigma_0^{\text{in}} = \Pi_0^{\text{in}}$. We then say that a formula is $\Sigma_\alpha^{\text{in}}$ if and only if it is of the form $\forall \exists \bar{x}_i \phi_i(\bar{x}_i, \bar{y})$, where each ϕ_i is Π_β^{in} for some $\beta < \alpha$. The Π steps of the hierarchy are built accordingly. We then define the Scott rank of a model M as the least ordinal α such that every automorphic orbit of a tuple of elements of M is definable with a $\Sigma_\alpha^{\text{in}}$ formula. Equivalently, by the result [2] of Montalbán, we can define it as the least ordinal α such that M has a Π_α^{in} Scott sentence. Additionally, for a theory T , one can consider its Scott spectrum, i.e. the set of ordinals α such that some $M \models T$ has Scott rank α .

In the work [3], Montalbán and Rossegger initiated the Scott analysis of models of Peano Arithmetic. They characterized the Scott spectra of completions of PA, showing in particular that the only model $M \models PA$ of finite Scott rank is the standard model. This was further extended in [1].

In a joint work with Leszek Kołodziejczyk and Mateusz Łełyk [5], we have conducted a similar analysis for subsystems of PA, in most cases ones extending $I\Delta_0 + exp$ such as $I\Sigma_n + exp$ or $B\Sigma_n + exp$. Among others, we have shown that for each n the Scott spectrum of $I\Sigma_n + exp + \neg B\Sigma_{n+1}$ consists of $n + 1$ and all countable ordinals from $n + 3$ onward. The methods used seem to fail irreparably in addressing the issue of models of $I\Sigma_n + exp + \neg B\Sigma_{n+1}$ of Scott rank $n + 2$.

A part of recent result by Wei Wang [6] about Kaye’s cardinality scheme for coded ultrapowers introduced by Paris [4], coupled with the refinements of previously obtained results and a particular construction of an ultrapower, allowed us to obtain a class of models of $I\Sigma_n + exp + \neg B\Sigma_{n+1}$, all with Scott rank $n + 2$. We will explore the proof, emphasizing yet another kind of construction and methods which can be used in the Scott analysis of weak systems of arithmetic.

[1] D. Gonzalez, M. Łełyk, D. Rossegger, and P. Szlufik, *Classifying the complexity of models of arithmetic*, 2025. arXiv 2507.12025 [math.LO].

[2] A. Montalbán, *A robust Scott rank*, Proceedings of the American Mathematical Society 143.12 (2015), pp. 5427–5436.

[3] A. Montalbán and D. Rossegger, *The structural complexity of models of arithmetic*, The Journal of Symbolic Logic 89.4 (2024), pp. 1703–1719.

[4] J. B. Paris, *Some conservation results for fragments of arithmetic*, Model

Theory and Arithmetic. Ed. by C. Berline, K. McAloon, and J.-P. Ressayre. Vol. 890. Lecture Notes in Mathematics. Berlin Springer, 1981, pp. 251–262.

[5] P. Szłufik, *Scott ranks of models of elementary arithmetic* (Sept. 10, 2025). Talk at JAF44, joint work with Leszek Kołodziejczyk and Mateusz Lelyk.

[6] W. Wang, *Finite Combinatorics and Fragments of Arithmetic*, 2025. arXiv 2506.17943 [math.LO].