

The cohesive and stable Ramsey theorems and proof size over a weak base theory

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Over the weak base theory RCA_0^* , most typical Ramsey-theoretic statements are $\forall\Pi_3^0$ -conservative, but they differ in terms of the effect on proof size for $\forall\Pi_3^0$ sentences. The usual Ramsey's theorem for pairs and two colours, RT_2^2 , induces iterated exponential proof speedup over RCA_0^* , even for very simple sentences [1]. In contrast, Kowalik [2] recently showed that the chain-antichain principle CAC can be polynomially simulated by RCA_0^* . Whether the combinatorial principles discussed above yield significant proof speedup over RCA_0^* depends on the kind of closure properties they imply for the definable cut I_1^0 , which is the intersection of all Σ_1^0 -definable cuts.

In the present work, we show that $\text{RCA}_0^* + \text{CRT}_2^2$, where CRT_2^2 is the cohesive Ramsey's theorem for pairs, implies exponential closure I_1^0 . Consequences include iterated exponential proof speedup of $\text{RCA}_0^* + \text{CRT}_2^2$ over RCA_0^* for Π_1 sentences and the unprovability of CRT_2^2 in $\text{RCA}_0^* + \text{CAC}$. On the other hand, we show that proofs of $\forall\Pi_3^0$ sentences in $\text{RCA}_0^* + \text{SRT}_2^2$, where SRT_2^2 is stable Ramsey's theorem for pairs, can be polynomially simulated by RCA_0^* . Nevertheless, SRT_2^2 also implies a nontrivial property of I_1^0 , specifically closure under functions of quasipolynomial growth rate.

This is a joint work with Leszek Kołodziejczyk.

[1] Kołodziejczyk, Leszek Aleksander and Wong, Tin Lok and Yokoyama, Keita, *Ramsey's theorem for pairs, collection, and proof size*, Journal of Mathematical Logic, vol. 24 (2024), n° 2, pp.2350007.

[2] Kowalik, Katarzyna W., *A non-speedup result for the chain-antichain principle over a weak base theory*, arXiv:2510.00323.