Kotlarski's question

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In the 1980s, Henryk Kotlarski published two papers devoted to the rich theory of elementary cuts in countable recursively saturated models of PA, [4, 5]. Let M be such a model. In both papers Kotlarski asked: What is the structure of the lattice of elementary substructures of M? In particular, does it depend on M?

For a model M, is the lattice of elementary submodels of M under inclusion, and $\mathrm{Lt}_0(\mathrm{M})$ is the $\vee$-subsemilattice of $\mathrm{Lt}(\mathrm{M})$ consisting of finitely generated elementary submodels of M. A partial answer to Kotlarski's question is the following theorem.

Theorem 1. ([2]) Let M and N be countable arithmetically saturated models of the same completion of PA. The following are equivalent:
(1) $\mathrm{M} \cong \mathrm{N}$;
(2) $\mathrm{Lt}(\mathrm{M}) \cong \mathrm{Lt}(\mathrm{N})$;
(3) $\mathrm{Lt}_0 \cong \mathrm{Lt}_0(\mathrm{N})$.

The proof of (3) $\to$ (1) is based on an intricate coding of the standard system of a countable arithmetically saturated model M of PA in $\mathrm{Lt}(\mathrm{M})$ using particular facts about finitely generated submodels of M. Arithmetic saturation plays a crucial role. It is open whether the result holds for recursively saturated models that are not arithmetically saturated. Many other open questions remain. I propose two generalizations of Kotlarski's question and I will present some partial answers based of some results from [3], [6], and the survey paper [1].

[1] Athar Abdul-Quader and Roman Kossak, *The lattice problem for models of PA*, Bull. Symb. Log., 32(2):259–286, 2026.

[2] Roman Kossak and James H. Schmerl, *Arithmetically saturated models of arithmetic*, Notre Dame J. Formal Logic, 36(4):531–546, 1995. Special Issue: Models of arithmetic.

[3] Roman Kossak and James H. Schmerl, *On cofinal submodels and elementary interstices*, Notre Dame J. Form. Log., 53(3):267–287, 2012.

[4] Henryk Kotlarski, *On elementary cuts in models of arithmetic*, Fund. Math., 115(1):27–31, 1983.

[5] Henryk Kotlarski, *On elementary cuts in recursively saturated models of Peano arithmetic*, Fund. Math., 120(3):205–222, 1984.

[6] James H. Schmerl, *The diversity of minimal cofinal extensions*, Notre Dame J. Form. Log., 63(4):493–514, 2022.