

Quasi-Polish spaces and spaces of unbounded filters in reverse mathematics

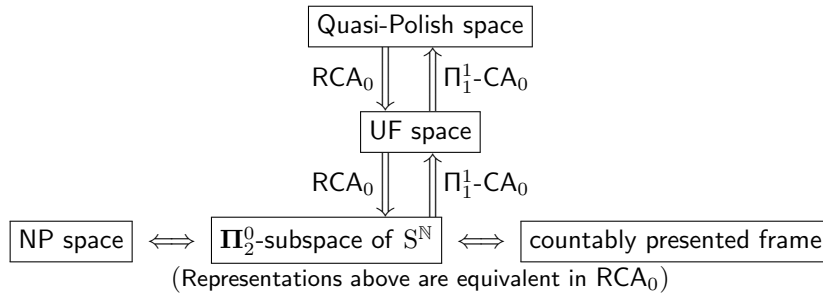
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The properties of separable complete metric spaces, also known as Polish spaces, have been well studied in reverse mathematics [1]. To study non-metrizable spaces in reverse mathematics, we need to formalize them in second-order arithmetic. In this talk, we formalize quasi-Polish spaces as a generalization of Polish spaces and study their properties.

In classical mathematics, quasi-Polish spaces provide a useful class when considering non-metrizable spaces, since every second-countable T_0 space is embeddable as a subspace of a quasi-Polish space. It is also known that quasi-Polish spaces are identical to Π_2^0 -subspaces of the countable power of the Sierpiński space $(S^{\mathbb{N}})$, as shown by de Brecht [2].

We first show that quasi-Polish spaces are equivalent to spaces of unbounded filters (UF spaces) and spaces of non-principal filters (NP spaces). This equivalence answers an open question from Mummert and Stephan [4], showing that the class of second-countable UF spaces is closed under countable products. Furthermore, these equivalences show why quasi-Polish spaces are important. For example, this class generalizes countable domains, making it important in computability theory. From the perspective of point-free topology, it can also be represented as the class of countably presented frames, as shown by Heckmann [5]. Moreover, in UF spaces, the reverse mathematical strength of classic topological theorems, such as Urysohn's metrization theorem, has been analyzed by Mummert [3].

We analyze the necessary and sufficient axioms to prove that a space under one representation is homeomorphic to the same space under a different representation. In other words, we show the reverse mathematical strength needed to switch between these different representations. As illustrated below, while some implications are provable over the base system RCA_0 , we show that the converses require the strong subsystem $\Pi_1^1\text{-CA}_0$



Besides the differences in space representations, there are two different ways

to represent elements in these spaces, and they have different logical strengths. For example, an element of a quasi-Polish space can be represented by either a left-Cauchy sequence or $\hat{d}$ -Cauchy sequence. In UF spaces, we can view a filter as a decreasing sequence or as a set. The second approach implies the first over RCA_0 , but the converse requires ACA_0 , as shown by Mummert [3]. This does not happen in metric spaces.

Furthermore, based on this setting, we can obtain some examples of reverse mathematical results on the classification of topological spaces, such as the Hurewicz theorem.

Stephen G. Simpson, *Subsystems of second order arithmetic*. Perspectives in Logic, second edition. Cambridge University Press, Cambridge, 2009.

[2] Matthew de Brecht, *Quasi-Polish spaces*, Ann. Pure Appl. Logic, 164(3): 356–381, 2013.

[3] Carl Mummer, *On the reverse mathematics of general topology*, The Pennsylvania State University, 2005.

[4] Carl Mummert and Frank Stephan, *Topological aspects of poset spaces*, Michigan Mathematical Journal, 59(1):3–24, 2010.

[5] Reinhold Heckmann, *Spatiality of countably presentable locales (proved with the Baire category theorem)*, Mathematical Structures in Computer Science, 25(7):1607–1625, 2015.