

# Undecidability and definability results using the coprime relation

Carl Jacobsen

We study undecidability and definability for certain extensions of  $\langle \mathbb{N}; \perp \rangle$ , where  $\perp$  denotes the coprime relation. This is done in first-order logic without a logical symbol for equality.

## Undecidability

Let  $S$  be the successor function. Woods proved in [3] that  $Th(\mathbb{N}; \perp, S)$  is undecidable. We generalize his arguments to show that, for an extension of  $\langle \mathbb{N}; \perp \rangle$ , a weak form of the independence property implies undecidability. Consequently, the following structures have undecidable theories:

- $\langle \mathbb{N}; \perp, n \mapsto |L(n)| \rangle$  for  $L \in \mathbb{Z}[X]$  not a monomial
- $\langle \mathbb{N}; \perp, n \mapsto a^n - 1 \rangle$  for  $a > 1$
- $\langle \mathbb{N}; \perp, n \mapsto \lfloor an/b \rfloor \rangle$  for  $a, b \in \mathbb{N}$  with  $a \perp b, b \geq 2$

## Definability

Define  $m \sim n$  if  $m$  and  $n$  have the same prime divisors. Another result of Woods [3] is that addition and multiplication are definable in  $\langle \mathbb{N}; \perp, S \rangle$  iff there is some  $k \in \mathbb{N}$  such that for all  $m, n \in \mathbb{N}$  we have

$$\left( \bigwedge_{i=0}^k S^i(m) \sim S^i(n) \right) \Rightarrow m = n.$$

The last claim is the *Erdős–Woods conjecture*. With  $m, n$  restricted to the set of prime powers  $\Pi$ , the conjecture has been proven. Using this Richard proved for any arithmetic  $A \subseteq \mathbb{N}^{\mathbb{N}}$  that  $A \cap \Pi^{\mathbb{N}}$  is definable in  $\langle \mathbb{N}; \perp, S \rangle$  [2].

The main theorem combines and generalizes these results. For certain extensions  $\mathcal{M}$  of  $\langle \mathbb{N}; \perp \rangle$ , it shows that an arithmetic relation is definable in  $\mathcal{M}$  iff it satisfies a number-theoretic condition.

Let  $A \subseteq \mathbb{N}^{\mathbb{N}}$  denote an arithmetic relation. We obtain the following, where the last result builds on Bès and Richard’s paper [1].

- $A$  is definable in  $\langle \mathbb{N}; \perp, S \rangle$  iff there is some  $k \in \mathbb{N}$  such that for all  $m \in A, n \in \mathbb{N}^{\mathbb{N}}$  we have

$$\left( \bigwedge_{j=1}^N \bigwedge_{i=0}^k S^i(m_j) \sim S^i(n_j) \right) \Rightarrow n \in A.$$

- Let  $\omega(n)$  be the number of prime divisors of  $n$ . Then  $A$  is definable in  $\langle \mathbb{N}; \perp, \omega \rangle$  iff for all  $m \in A, n \in \mathbb{N}^{\mathbb{N}}$  we have

$$\left( \bigwedge_{j=1}^N m_j \sim n_j \right) \Rightarrow n \in A.$$

---

• For  $a \geq 3$  let  $P_a$  be the set of primes and  $a$ -th powers of primes and  $<_{P_a}$  be the restriction of  $<$  to  $P_a$ . Then  $A \cap P_a^N$  is definable in  $\langle \mathbb{N}; \perp, <_{P_a} \rangle$ .

[1] Alexis Bès and Denis Richard, *Undecidable extensions of Skolem arithmetic*, The Journal of Symbolic Logic, 63(2):379–401, 1998.

[2] Denis Richard, *All arithmetical sets of powers of primes are first-order definable in terms of the successor function and the coprimeness predicate*, Discrete Mathematics, 53:221–247, 1985.

[3] Alan Woods, *Some problems in logic and number theory and their connections*, Ph. D. Thesis, Univ. of Manchester, 1981. First publication in New Studies in Weak Arithmetics, CSLI Publications, Stanford, Lecture Notes 211, 2013, xiv + 448 p.