

Unfolding of Grzegorczyk systems

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Feferman's unfolding program asks, given a schematic system S , which operations, predicates, and principles concerning them ought to be accepted once one has accepted S (Feferman, [3]). So far, the program has investigated several foundational stances. These include but are not limited to the unfolding of non-finitist arithmetic $\mathcal{U}(\text{NFA})$ (Feferman and Strahm, [4]), the unfolding of finitist arithmetic $\mathcal{U}(\text{FA})$ (Feferman and Strahm [5]), and the unfolding of feasible arithmetic $\mathcal{U}(\text{FEA})$ (Eberhard and Strahm, [2]). This investigation yielded the following technical results: the full unfolding of NFA is proof-theoretically equivalent to predicative analysis $\text{RA}_{<\Gamma_0}$; the full unfolding of FA is proof-theoretically equivalent to primitive recursive arithmetic PRA; the provably total functions of the operational, the intermediate, and the full unfolding of FEA are exactly the polynomial-time computable functions.

These results occupy the predicative, finitist, and feasible limits, respectively. Between the feasible and finitist limits, however, the unfolding spectrum remains unexplored. This invites the investigation of the unfolding of a sequence of schematic systems G_n corresponding to the levels of the Grzegorczyk hierarchy. The Grzegorczyk hierarchy is the ideal candidate to fill this space since the union of its levels $\mathcal{E}^n$ exhausts the primitive recursive functions while maintaining strict inclusion $\mathcal{E}^n \subsetneq \mathcal{E}^{n+1}$ at every stage.

This work develops the unfolding of schematic systems G_n over the natural numbers, utilizing the methodology of Eberhard and Strahm, [2], but equipped with a growing $\mathcal{L}_n = \{E_1, \dots, E_n\}$ whose function symbols denote the first n hyperoperations, alongside open-ended successor induction. The reason for the choice of this formalization is to move the bounded totality requirement of FEA from binary words onto a unary successor representation. Under length induction along binary words, reaching an input β costs roughly $\log \beta$ steps, meaning iterated composition of a step function remains computationally low. Under a unary successor, however, it costs β steps, and β -fold composition is the mechanism allowing the ascension of the hierarchy levels. Here, the computational level is not rigidly stipulated but dynamically emerges. By accepting G_n , the unfolding process accommodates closure under E_n -bounded recursion, with the finite base language capping the available bounding terms.

The central philosophical claim of this work rests on the notion of totality in our framework. The base theory of the unfolding is a partial combinatory algebra (PCA) which provides a general fixed point combinator. Using open-ended successor induction, the operational unfolding of G_n , $\mathcal{U}_0(G_n)$, can prove that the fixed-point term for E_{n+1} converges on every input. But the system cannot bound E_{n+1} by any $\mathcal{L}_n$, since E_{n+1} strictly dominates every composition of $E_1, \dots, E_n$. Consequently, dropping the bounding requirement collapses the hierarchy and the provably convergent functions of $\mathcal{U}_0(G_0)$ encompass all of PRA uniformly for any n . In this system the commitments are tied to the

intension of “total”.

This offers a new insight into the debate on the implicit commitment thesis (ICT), especially on the notion of epistemic stability introduced by Dean, [1]. ICT is mostly discussed in the context of meta-mathematical reflection and consistency (Łelyk and Nicolai, [7]; Głowacki and Zicchetti, [6]), whereas we investigate it through the notion of unfolding of an open-ended schematic system. In our proposed framework, epistemic stability is achieved by a strict notion of bounded totality, while an unbounded notion of totality leads to an epistemically unstable theory. We do not claim totality as the source of stability in general; stability is relative to the notion of commitment a stance is tested against, and reflection-based notions yield their own boundaries. But for stances whose formal explication is a closure condition on functions the constitutively appropriate notion of commitment is totality.

[1] Dean, Walter (2015), *Arithmetical Reflection and the Provability of Soundness*, *Philosophia Mathematica* 23.1, pp. 31–64. doi: 10.1093/phimat/nku026.

[2] Eberhard, Sebastian and Thomas Strahm (2015), *Unfolding Feasible Arithmetic and Weak Truth*, *Unifying the philosophy of truth*, Springer, pp. 153–167.

[3] Feferman, Solomon (1996), *Gödel’s program for new axioms: Why, where, how and what?*, Gödel ’96. Ed. by P. Hájek. Vol. 6. *Lecture Notes in Logic*, Springer, pp. 3–22.

[4] Feferman, Solomon and Thomas Strahm (2000), *The unfolding of non-finitist arithmetic*, *Annals of Pure and Applied Logic* 104.1-3, pp. 75–96. doi: 10.1016/S0168-0072(00)00008-7.

[5] — (2010), *Unfolding Finitist Arithmetic*, *The Review of Symbolic Logic* 3.4, pp. 665–689. doi: 10.1017/S1755020310000183.

[6] Głowacki, Maciej and Matteo Zicchetti (2025), *Implicit Commitments, Epistemic Stability, and the Acceptability of Consistency*, *Erkenntnis*, pp. 1–24.

[7] Łelyk, Mateusz and Carlo Nicolai (2022), *A theory of implicit commitment*, *Synthese* 200.4, p. 284.