Scott ranks of models of elementary arithmetic

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$L_{\omega_1,\omega}$ extends first-order logic with countable connectives, i.e. if $\bar{x}$ is a finite tuple of variables and $\varphi_0(\bar{x}), \varphi_1(\bar{x}), \ldots$ are formulae then $\bigvee_{i\in\omega} \varphi_i(\bar{x})$ and $\bigwedge_{i\in\omega} \varphi_i(\bar{x})$ are formulae as well.

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Theorem (Scott, '65)

For every countable structure M, there is an $L_{\omega_1,\omega}$-sentence φ_M such that countable structures satisfying φ_M are exactly isomorphic copies of M.

Any such φ_M would be further called a *Scott sentence* of M.

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Scott rank

Definition (Montalbán, 2015)

Scott rank of a countable first-order structure M, denoted $SR(M)$, is the least ordinal α s.t. M has a Scott sentence $\varphi \in \Pi^{in}_{\alpha+1}$; equivalently, every automorphic orbit of any tuple of elements of M is definable with a Σ^{in}_{α} formula.

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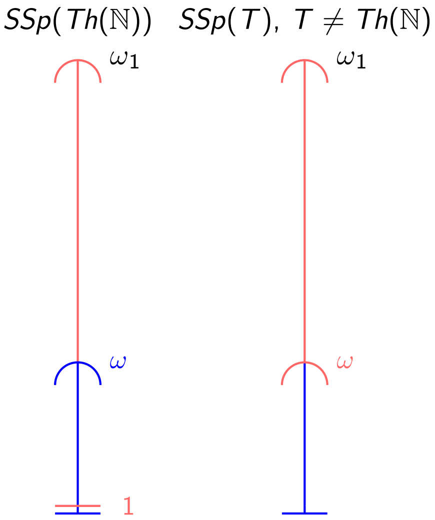
Definition

Given a first-order theory T, *Scott spectrum* of T is the set

$$SSp(T) = \{\alpha < \omega_1 \mid SR(M) = \alpha \text{ for some } M \models T\}.$$

Scott analysis of PA



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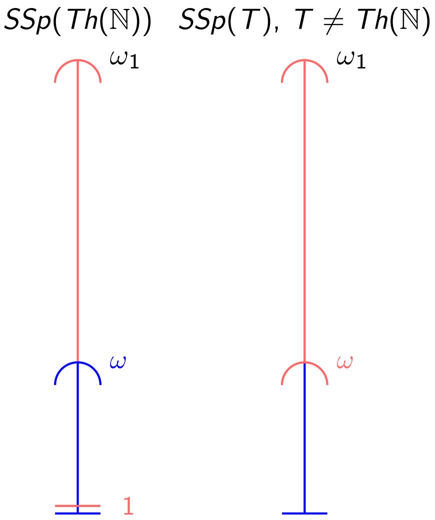

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All nonstandard nonatomic models of PA have Scott ranks $\geq \omega + 1$.

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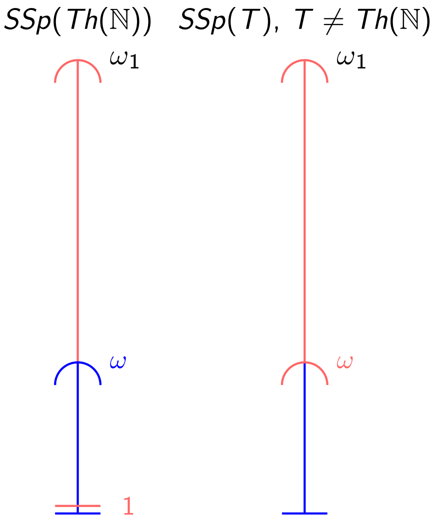

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Corollary (GŁRS)

All recursively saturated models of PA have Scott ranks $\omega + 1$.

Results for models of induction

Theorem

For $n \geq 1$ and $M \models I\Sigma_n + \neg B\Sigma_{n+1}$ we have that $SR(M) \geq n+1$ and the bound is tight.

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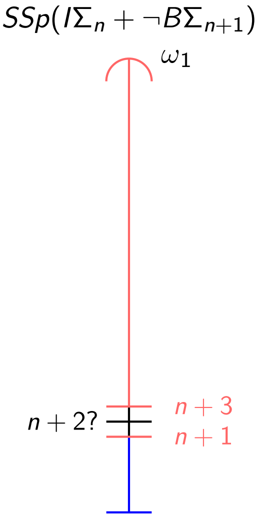

Results for models of collection

Theorem

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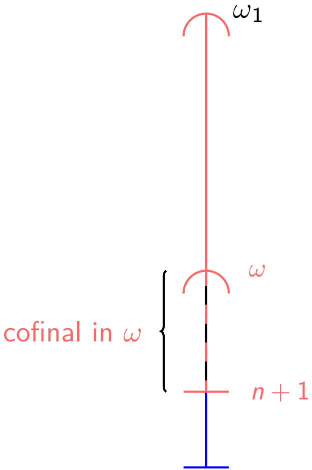
Theorem

For every every n and every ordinal $\alpha \geq 1$ there is a model $M \models B\Sigma_n + \neg I\Sigma_n$ with $SR(M) = n + \alpha$ or $SR(M) = n + 1 + \alpha$.

Results for models of collection

$SSp(B\Sigma_n + \neg I\Sigma_{n+1})$



Results regarding recursive saturation

Definition

We say that a type $p(\bar{x}, \bar{a})$ over M is computable iff the set $\{\ulcorner \varphi(\bar{x}, \bar{y}) \urcorner \mid \varphi(\bar{x}, \bar{a}) \in p(\bar{x}, \bar{a})\}$ is computable.

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Theorem

Every recursively saturated model $M \models EA$ has Scott rank $\omega + 1$.

Key lemma

Lemma (Expressivity collapse)

For every $n \geq 1$, every Σ_n-recursively saturated model $M \models EA$ and $a, b \in M$ the following equivalence holds:

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Question: Can we find more (interesting) kinds of models of (extensions of) PA^- such that the equivalence still holds?

Lower bound – proof outline with *exp*

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For $n \geq 1$ and $M \models B\Sigma_n + exp$ we have that $SR(M) \geq n + 1$.

Draft of the proof.

WLOG (really!) there are nonstandard $a, b \in M$ s.t. orbit of a is a subset of $[0, b]$.

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By standard arguments we can take $e < a$ coding Π_{n-1}-type of $\langle c, d \rangle$ and the least $\langle c_0, d_0 \rangle$ with its Π_{n-1}-type coded by e.

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By EC lemma, $\langle c, d \rangle$ and $\langle c_0, d_0 \rangle$ have the same Π_{n-1}^{in}-type and so d_0 witnessess the definition of automorphic orbit of c.

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c was arbitrary, hence every element of M is Σ_n-definable with parameters below b, which we can use to create Σ_n multifunction violating PHP.

What's next?

Question 1.
How do finite part of the spectra of $I\Sigma_n + \neg B\Sigma_{n+1}$ and $B\Sigma_n + \neg I\Sigma_n$ look like?

Question 2.
Can a completion of $I\Delta_0 + exp$ omit Scott rank ω?

Question 3.
What are possible Scott ranks of models of $I\Delta_0 + \neg exp$? What about $IOpen + \neg I\Delta_0$ $(+\neg exp)$?

Scott spectrum for I-models – proof outline for finite part

Theorem

For every n and every finite $k \geq 1$ there is a model $K \models I\Sigma_n + \neg B\Sigma_{n+1}$ with $SR(K) = n + 2 + k$.

Start with a model M' of PA and take $M = K^{k+1}(M')$.
Using the ACT and LL_1-basis theorem, there is N such that $M \triangleright N \models I\Sigma_n + exp$ with LL_1-definable satisfaction predicate S_N for N.
Take $K \subseteq N$ consisting of Σ_{n+1}-M-definable elements.
Canonically embed M into K (via successor) and K into N (via definitions).
Repeat the construction of K in the image M^* of the first embedding.
Upper bound on Scott rank of K comes from careful computing of complexities of definitions of elements.
Lower bound comes from observation, that ω cannot be Σ_{n+k+1}-definable in K since it is not Σ_{k+1}-definable in M.

Scott spectrum – proof outline for infinite part

Theorem

For every infinite $\alpha \geq \omega$ there is a model $M \models I\Sigma_n + \neg B\Sigma_{n+1}$ and model $N \models B\Sigma_n + \neg I\Sigma_n$ with $SR(M) = SR(N) = \alpha$.

Start with a model M' of PA with $SR(M') = \alpha$.

Essentially by a construction from before (even more careless) you can arrive at M and N as desired, both biinterpretable with M'.

Now all formulae used in these interpretations are Σ_m for some $m \in \omega$.

It follows that
$SR(M) \leq m + SR(M') = SR(M') \leq m + SR(M)$ it follows that $SR(M)$ is infinite and thus $SR(M') = SR(M)$. Similarily for N.

Key lemma – proof outline

Lemma

For every $n \geq 1$, every Σ_n-recursively saturated model $M \models EA$ and $a, b \in M$ the following equivalence holds:

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Consider true disjunct $\varphi(a, c)$ and argue that by equality of Σ_{k+1} types, there is d such that $\varphi(b, d)$ is true as well.

Conclude that $\Phi(b)$ holds.

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Lower bound – proof outline for $n = 1$ with *exp*

Theorem

For $M \models B\Sigma_1 + exp$ we have that $SR(M) \geq 2$.

Draft of the proof.

Assume the contrary. It follows that for every element its orbit is Σ_1^{in} definable and so Σ_1 definable.

Fix $c \in M$ and assume that its orbit is defined by $\exists \bar{y} \delta(x, \bar{y})$.

Note that there is c_0 in the automorphic orbit of c, which is Σ_1-definable, namely by „the projection onto the first coordinate of the smallest pair $\langle x, y \rangle$ such that $\delta(x, y)$ holds"

Deduce that every automorphic orbit is singleton.

Recall that no pointwise Σ_1 definable model of *EA* satisfies $B\Sigma_1$.