# On the finite cohesiveness principle

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September 10, 2025

# Reverse math

- Reverse mathematics is a program that seeks to determine which axioms are required to prove theorems of mathematics.
- The **Big Five**:

$$\mathrm{RCA}_0 < \mathrm{WKL}_0 < \mathrm{ACA}_0 < \mathrm{ATR}_0 < \Pi^1_1\text{-}\mathrm{CA}_0.$$

- Most mathematic theorems (that can be formalized in second-order arithmetic) are proven to be equivalent to one of the above systems.
- However, some theorems (e.g., $\mathrm{RT}^2_2$) are exceptions...

# Fragments of first- and second-order arithmetic

- The language of first/second-order arithmetic: $\mathcal{L}_1 = \{+, \times, <, =, 0, 1\}$, $\mathcal{L}_2 = \{+, \times, <, =, 0, 1, \in\}$.
- $\mathrm{I}\Sigma^0_n$ consists of $\mathrm{PA}^-$ and **Induction** for all $\Sigma^0_n$ formulas $\varphi$:

$$\varphi(0, \bar{c}) \wedge (\forall x\ (\varphi(x, \bar{c}) \to \varphi(x+1, \bar{c})) \to \forall x\ \varphi(x, \bar{c}).$$

- $\mathrm{B}\Sigma^0_n$ consists of $\mathrm{I}\Delta^0_0$ and **Collection** for all $\Sigma^0_n$ formulas $\varphi$:

$$\forall x < a\ \exists y\ \varphi(x, y, \bar{c}) \to \exists b\ \forall x < a\ \exists y < b\ \varphi(x, y, \bar{c}).$$

- $\exp$ denotes the totality of the exponentiation function.
- (Paris–Kirby 1978) $\mathrm{I}\Delta^0_0 + \exp \dashv \mathrm{B}\Sigma^0_1 + \exp \dashv \mathrm{I}\Sigma^0_1 \dashv \mathrm{B}\Sigma^0_2 \dashv \mathrm{I}\Sigma^0_2 \dots$ and none of the converses holds.
- $\mathrm{RCA}_0 = \mathrm{I}\Sigma^0_1 + \Delta^0_1$-comprehension. $\mathrm{WKL}_0 = \mathrm{RCA}_0+$"each infinite binary tree has an infinite path".

# Cohesiveness principle

## Definition

- Let $R = (R_k)_{k \in \mathbb{N}}$ be a sequence of sets. We say that a set $H$ is $R$-**cohesive** if for any $k \in \mathbb{N}$, either $H \subseteq^* R_k$ or $H \subseteq^* R_k^c$ ($A \subseteq^* B$ means $A \subseteq B$ except finitely many elements)
- The cohesiveness principle $(\mathrm{COH})$ claims that every sequence of sets has an infinite cohesive set.

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## Facts

- (Cholak–Jockusch–Slaman 2001, Mileti 2004) $\mathrm{RCA}_0 \vdash \mathrm{RT}^2_2 \leftrightarrow \mathrm{SRT}^2_2 + \mathrm{COH}$.
- (CJS 2001, Chong–Slaman–Yang 2012, Belanger 2015) $\mathrm{COH}$ is $\Pi^1_1$-conservative over $\mathrm{RCA}_0$, $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$, $\mathrm{RCA}_0 + \mathrm{I}\Sigma^0_2 \ldots$

# A weaker base theory: $\mathrm{RCA}_0^*$

### Definition (Simpson–Smith 1986)

The system $\mathrm{RCA}_0^*$ consists of $\mathrm{B}\Sigma_1^0$, $\Delta_1^0$-comprehension and $\exp$.

### Facts

Let $(M, \mathcal{X}) \models \mathrm{RCA}_0^* + \neg\mathrm{I}\Sigma_1^0$, then

- there is a proper cut $I$ and a monotone function $G \colon I \to M$ in $\mathcal{X}$, whose range is cofinal in $M$.
- (Chong–Mourad 1990) Let $X \subseteq I$. If $X$ and $I \setminus X$ are both $\Sigma_1^0$-definable, then $X \in \mathrm{Cod}(M/I)$.

# Ramsey over $\mathrm{RCA}_0^*$

## Facts/Theorems (Fiori-Carones–Kołodziejczyk–Kowalik 2020)

- Many Ramsey-style principles are mutually transferred between the model and the cut.

$$(M, \mathcal{X}) \models P \iff (I, \mathrm{Cod}(M/I)) \models P,$$

where $P \in \{\mathrm{CAC}, \mathrm{ADS}, \mathrm{RT}^2_2, \mathrm{RT}^3_2 \dots\}$.
- As a consequence, the strength of Ramsey style theorems over $\mathrm{RCA}_0^*$ differs from that over $\mathrm{RCA}_0$.
- However, $\mathrm{COH}$ seems to be an exception.

## Question (FKK 2020, Belanger 2015)

Is $\mathrm{RCA}_0^* + \neg \mathrm{I}\Sigma^0_1 + \mathrm{COH}$ consistent? Is there any conservation results for $\mathrm{COH}$ over $\mathrm{RCA}_0^*$?

# Cohesiveness principle vs $\mathrm{RCA}_0^*$

**Theorem (S. 2025)**

$\mathrm{RCA}_0^* + \mathrm{COH} \vdash \mathrm{I}\Sigma_1^0$.

# Cohesiveness principle vs $\mathrm{RCA}_0^*$

## Theorem (S. 2025)

$\mathrm{RCA}_0^* + \mathrm{COH} \vdash \mathrm{I}\Sigma_1^0$.

## Proof

Suppose not, fix a model $(M, \mathcal{X}) \models \mathrm{RCA}_0^* + \neg\mathrm{I}\Sigma_1^0 + \mathrm{COH}$.
Let $I$ and $G\colon I \to M$ be as above.

- Consider the following instance of $\mathrm{COH}$:

$$(R_x)_{x\in M} = \{\langle n, x\rangle \in M \mid \exists i \in x \; n \in [G(i), G(i+1))\} \in \mathcal{X}$$

- Let $H$ be $(R_x)_{x\in M}$-cohesive. The set below is $\mathrm{Cod}(M/I)$-cohesive in $(I, \mathrm{Cod}(M/I))$.

$$H' = \{i \in I \mid \exists n \in H \; n \in [G(i), G(i+1))\}$$

- By the Chong–Mourad coding lemma, $H'$ itself is also in $\mathrm{Cod}(M/I)$, contradiction! □

Remark: $H$ being $(R_x)_{x<2^b}$-cohesive suffices, assuming $b > I$.

# Finite cohesiveness principle

## Definition

The finite cohesiveness principle ($\text{fin-COH}$) claims that every finite sequence of sets has an infinite cohesive set.

## Question

How strong is $\text{fin-COH}$?

## Facts

- $\mathrm{RCA}_0^* + \text{fin-COH} \vdash \mathrm{I}\Sigma_1^0$.
- $\text{fin-COH}$ is $\Pi_1^1$-conservative over $\mathrm{RCA}_0$.

# The strength of $\text{fin-COH}$

With strong enough induction, $\text{fin-COH}$ becomes trivial.

**Theorem**

$\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2 \vdash \text{fin-COH}$.

**Proof.**

Let $R = (R_i)_{i<b}$ be a sequence of sets of length $b$. Then

$$\mathbb{N} = \bigcup_{\mathsf{len}(\sigma)=b} \bigcap_{i<b} R_i^{\sigma(i)}$$

where $R_i^0 = R, R_i^1 = R^c$.

By $\mathrm{B}\Sigma^0_2$, $\bigcap_{i<b} R_i^{\sigma(i)}$ is infinite for some $\sigma$, which is trivially $R$-cohesive. □

# The strength of $\mathrm{fin\text{-}COH}$

However(fortunately?), $\mathrm{fin\text{-}COH}$ is not completely trivial.

Theorem (S. 2025)

$\mathrm{WKL}_0 \nvdash \mathrm{fin\text{-}COH}$.

We postpone the proof a few slides later.

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Does $\mathrm{RCA}_0 + \neg \mathrm{B}\Sigma^0_2 \vdash \mathrm{fin\text{-}COH} \leftrightarrow \mathrm{COH}$?

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Proposition (S. 2025)

$\mathrm{RCA}_0 +$ "There is a $\Sigma^0_2$ partial **surjection** with bounded domain" implies $\mathrm{fin}\text{-}\mathrm{COH} \leftrightarrow \mathrm{COH}$.

# The strength of fin-COH

## Proposition (S. 2025)

$\mathrm{RCA}_0$ + "There is a $\Sigma^0_2$ partial **surjection** with bounded domain" implies $\text{fin-COH} \leftrightarrow \text{COH}$.

## Observation

Let $R = (R_i)$, $S = (S_j)$ be two sequences of sets. If for any $i$ there is some $j$ such that $R_i =^* S_j$, then any $S$-cohesive set $H$ will also be $R$-cohesive.

## Proof.

- Let $f\colon a \to \mathbb{N}$ be such a surjection. Take $g\colon a \times \mathbb{N} \to \mathbb{N}$ as its approximation. i.e., $\forall j < a \ f(j) = \lim_{x\to\infty} g(j,x)$.
- For any sequence of sets $R = (R_i)_{i\in\mathbb{N}}$, define $(S_j)_{j<a}$ by

$$x \in S_j \iff x \in R_{g(j,x)}.$$

- Now $S_{f^{-1}(i)} =^* R_i$ for any $i \in \mathbb{N}$.

□

# The strength of fin-COH

**Theorem**

$\mathrm{WKL}_0 \nvdash \text{fin-COH}$.

**Proof.**

- There exists a (non-standard) pointwise $\Sigma_2$-definable model of $\mathrm{I}\Sigma_1$, which satisfies the assumption in the proposition above.
- (Cholak–Jockusch–Slaman 2001) There is an $\omega$-model satisfying $\mathrm{WKL}_0 + \neg\mathrm{COH}$.
  - There is a $\omega$-model of $\mathrm{WKL}_0$ consisting only of low sets.
  - (Jockusch–Stephan 1993) Any set cohesive for all the primitive recursive sets cannot be low.
- The construction and the proof above can be carried out in models of $\mathrm{I}\Sigma_1$.

□

# Summary

- $\mathrm{RCA}_0^* + \text{fin-COH} \vdash \mathrm{I}\Sigma_1^0$.
- $\mathrm{RCA}_0 + \mathrm{B}\Sigma_2^0 \vdash \text{fin-COH}$.
- $\mathrm{WKL}_0 \nvdash \text{fin-COH}$.
- $\mathrm{RCA}_0 +$
  "There is a $\Sigma_2^0$ partial **surjection** with bounded domain"
  implies $\text{fin-COH} \leftrightarrow \mathrm{COH}$.

**Thank You!**