Definability Lattices for Very Weak Arithmetics

New Trends and Open Problems in Definability Theory

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Journées sur les Arithmétiques Faibles
Weak Arithmetics Days 44
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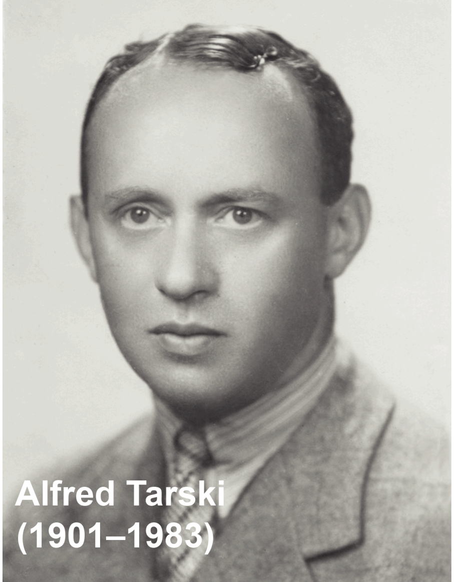
Alfred Tarski



«Mathematicians, in general, do not like to operate with the notion of definability; their attitude towards this notion is one of distrust and reserve.»

— Alfred Tarski, "On definable sets of real numbers", 1931

Alfred Tarski

Our goal is to continue changing this attitude:

- to attract attention to the field of research on definability,
- to provide examples and instruments
- to start composing a list of open problems, and solving them.

The basic notions of mathematical logic and mathematics

- Truth
- Proof
- Computation
- Definition

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- Truth
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The problem to explain something to somebody, to define something through something is a very existential one.

Definitions of Mathematical structures. Geometry and Arithmetic

Italy:

Giuseppe Peano,
Alessandro Padoa,
Mario Pieri, . . .

Giuseppe Peano
1858 – 1932

Mario Pieri
1860 – 1913

Alessandro Padoa
1868 – 1937

Germany:

Moritz Pasch,
Gotlob Frege,
David Hilbert, . . .

Moritz Pasch
1843-1930

Gotlob Frege
1848-1925

David Hilbert
1862 - 1943

Target: to find ”the best” set of primitive notions for Geometry and Arithmetic considered as deductive systems.

The main notion(s) of definability theory

- Domain (universe) U.
- Logical language L: the basic case — Logic of Relations = First-order predicate logic without functional symbols.
- R — a relation, S — a set of relations.
- R is definable through S iff for some formula in L we can interpret its relational (predicate) symbols (names) as relations from S and get R.
- Closure of S — all definable
- Closed set — definability space (REDUCT).
- Partial order on spaces, definability (complete) lattice.
- Definability lattice of a structure — all subspaces of its space.

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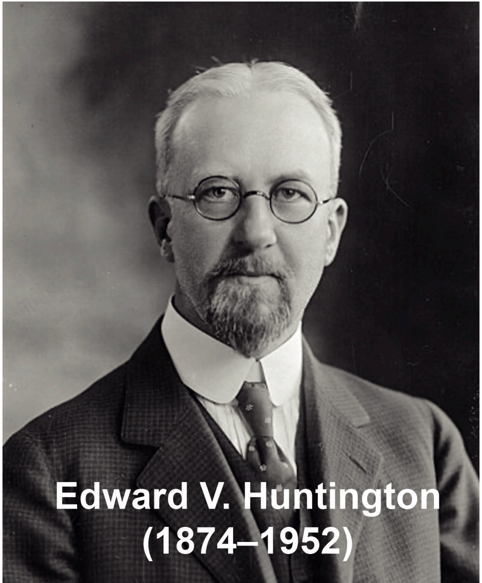
The definitions are invariant to choice of a signature of the structure — 'coordinate-free'.

Huntington. The Basic Example



Edward Huntington (1906–1935) described subspaces of (linear) order:

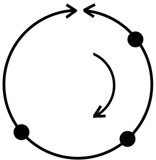
$x_1 < x_2$ — order;

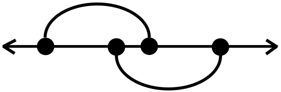
$(x_1 < x_2 < x_3) \vee (x_3 < x_2 < x_1)$, — "Between";

$(x_1 < x_2 < x_3) \vee (x_2 < x_3 < x_1) \vee (x_3 < x_1 < x_2)$ — "Cycle" clockwise;

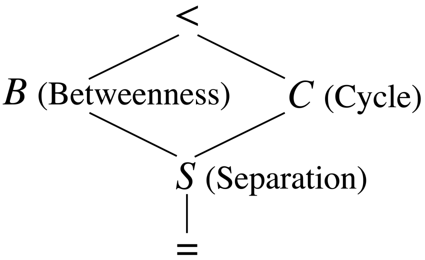
— "Separation": segments with endpoints x_1, x_3 and endpoints x_2, x_4 intersect, but are not included one to another,

— equality.





The lattice for rational order — the first



Claude Frasnay (1965) proved that these 5 elements exhaust the lattice of $\langle \mathbb{Q}; < \rangle$.

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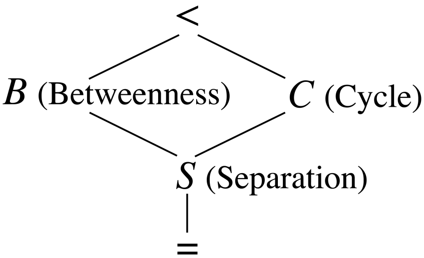

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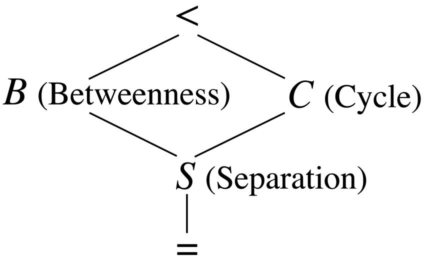

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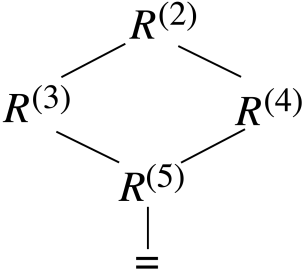
Teaser: Let us add to $\langle \mathbb{Q}; < \rangle$ one constant — 0. How many spaces shell we get? Estimate the figure.

Homogeneity — the critical factor

$\langle \mathbb{Q}; < \rangle$ — homogeneous = each (partial) automorphism of finite (sub)structures extends to an automorphism of the whole. It implies ω-categoricity.

Simon Thomas (1991) – another famous homogeneous structure:
the random (universal, generic) graph (undirected). For $k = 2, 3, 4, 5$, $R^{(k)}$ = all k-tuples of elements $x_1, \dots, x_k$ pairwise distinct with odd number of edges between them, $R^{(2)} = R$ the graph relation.



Finite latticies for homogeneous structures

Numerous results for homogeneous structures (Peter Cameron, Alistair Lachlan, Csaba Szabó).

Let us add a constant

Junker – Ziegler rational order with a cut:

- rational (0) — 116.
- irrational — 53.

András Pongrácz: «A negative “result” is the random graph with a fixed constant, on which the author together with two collaborators, gave up after having found 300 reducts.»

Thomas Conjecture

Every countable homogeneous structure with a finite relational language has finite definability lattice.

Simon Thomas

Bio:

"In 1973, Simon Thomas was the Welsh Youth Champion in the 800 metres. Since then, his life has gone steadily downhill."

Ph.D. University of London 1983.

Other Rutgers Logic Faculty: Gregory Cherlin, Dima Sinapova, Filippo Calderoni, Saharon Shelah. (Gelfand...)

NON-definability

How to prove NON-definability = impossibility to define?

- Impossibility in Mathematics: Galois — Abel etc.
- Automorphisms = transformations — invariants = symmetries

Felix Klein Erlangen Program

'Geometry of Logic'

Definability space $S \mapsto$ Group of its automorphisms Γ_S

Galois connection — (anti)-homomorphism between the definability lattice of a structure and

the lattice of clossed supergroups of the automorphism group of the structure.

Automorphisms

$\Gamma_S \not\subset \Gamma_{\{R\}} \Rightarrow R \notin [S]$

ω-categorical structures

Theorem of isomorphism

ω-categorical case [Ryll-Nardzewski, Engeler and Svenonius, end of 1950-s]

For every ω-categorical structure its lattice is

(anti)-Isomorphic to the lattice of closed supergroups of the automorphism group of the structure.

S — definability space, R — relation

Automorphisms

$\Gamma_S \not\subset \Gamma_{\{R\}} \iff R \notin [S]$

Examples of automorphism (transformation) groups

- Rational order: inversion, interval exchange
- Random graph: complement, complement except one vertex

What is happening beyond homogeneity?

Homogeneous structures are 'quantifier-flat'

- Space of every homogeneous structure is finitely generated in quantifier-free (Boolean) logic
- (quantifier) height of a space — minimal number of quantifier alternation to obtain the closure starting with a finite set of relations

Open problem 1

(Junker – Ziegler) is 0 height invariant?

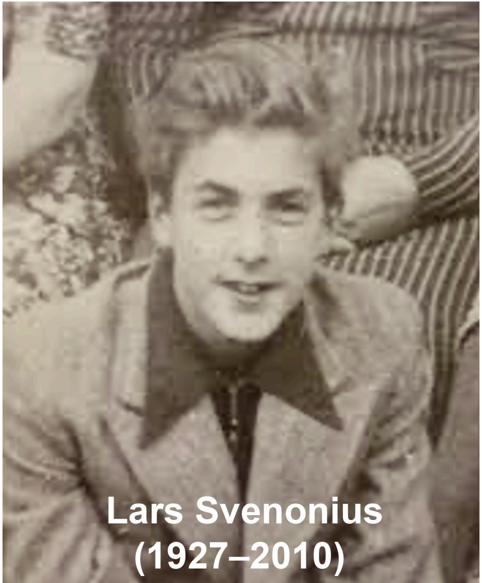
Limited height Th. (SS)

Every finite height is possible.

Open problem 2

To find 'natural' examples for height 2, 3,...

Lars Svenonius: Completeness theorem for Geometry of Logic



Automorphisms

$\Gamma_S \not\subset \Gamma_{\{R\}} \Rightarrow R \notin [S]$

Can we achieve equivalence?

Svenonious theorem,1959

For every relational structure, every subspace S of its definability lattice and every relation $R \notin S$ there is an elementary extension of the structure for which $\Gamma_S \not\subset \Gamma_{\{R\}}$.

— to add 'imaginary' elements.

Definability started with geometry.

Lars Svenonius: Completeness theorem for Geometry of Logic

Observer upon reading the papers of Tarski, might have wondered about the existence of general theorems which would explain elementary definability as the above theorems [explain] the basic properties of elementary logical consequence.

With the appearance of Klein's Erlanger Programm in 1872, it became apparent that automorphism groups are most useful means of studying mathematical theories.

...logicians seem slow in recognizing Svenonius' theorem (1959) as a basic tool in the theory of definability.

— Richard Buchi, Kenneth Danhof 1973, "Definibility in normal theories"

The simplest case outside homogeneity is $\langle \mathbb{Z}; +1 \rangle$

Integers with successor

Semenov, Soprunov. "The Lattice of Definability (of Reducts) for Integers with Successor." Izv. Math. 85 (2021).

D_n 'to be in distance n' ; E_n of 'n-codirectional' or 'n-equipollence' of segments:

$$D_n(a,b) \rightleftharpoons |a-b| = n$$

$$E_n(a,b,c,d) \rightleftharpoons a-b=c-d=n \vee a-b=c-d=-n$$

definability lattice for $\langle \mathbb{Z}, +1 \rangle$ consists of the spaces of =, $+n, D_n, E_n$, for all positive integers n. All lattice operations are costructively described.

Immediate:

Open problem 3

Describe the definability lattice for $\langle \mathbb{N}, +1 \rangle$.

All finite sets are definable!

Upward complete structures

A countable structure $\mathscr{M}$ is called *upward complete* , if $\mathscr{M} \cong \mathscr{M}'$ for every countable elementary extension of $\mathscr{M}' \succ \mathscr{M}$. An upward complete structure elementary equivalent to a structure is called completion of the latter [2].

For upward complete structures, the Svenonius theorem has a simple form:

Theorem

Let S^-, S be countable definability spaces on a universe A and $S^- \subset S$, the structure with the universe A and space S is upward complete. Then the following (a) and (b) are equivalent for every relation $R \in S$.

(a) $R \in S^-$.

(b) The group of permutations on A, preserving S^-, preserves R.

Semenov, Soprunov, Automorphisms and definability (of reducts) for upward complete structures. Mathematics, 10(20), 3748, (2022).

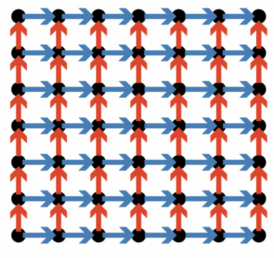
Discretely homogeneous graphs

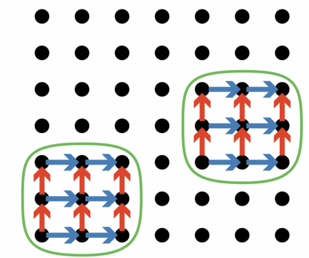
A graph G is *discretely homogeneous* iff it is connected and for every radius r, all centered r-neighbourhoods, except of finitely many, are isomorphic.

We proved that every discretely homogeneous structure has a complete upward extension with a fairly simple automorphism group.

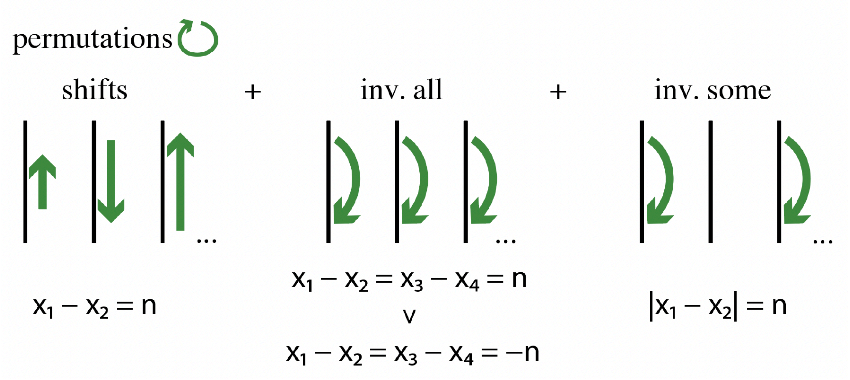
Two commutative successors on $\mathbb{Z}^2$

2D infinite checkered paper.





$\langle \mathbb{Z}; +1 \rangle$ is upward complete and discretely homogeneous



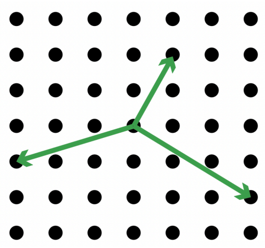
All residual classes modulo n are shifted independently by multiples of n and permute.

An Open Problem. Checkered Paper

Open problem 4

Describe the definability lattice for two commutative successors on $\mathbb{Z}^2$

A special case: spaces, generated by disjunctions of 2-relations — shifts

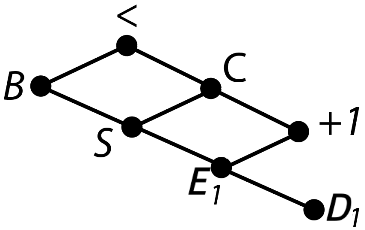


The simplest case: neibourghs — D_1 on $\mathbb{Z}^2$.

A next step is $\langle \mathbb{Z}; < \rangle$

A natural next step is $\langle \mathbb{Z}, \{<\} \rangle$. There are spaces B, C, S, all different.

We can prove that the definability lattice inbetween < and D_1 is



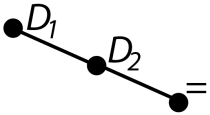
Fragment of $\langle \mathbb{Z}, \{<\} \rangle$ lattice

Below +1 we have $+n, D_n, E_n$, = as in $\langle \mathbb{Z}, +1 \rangle$.

Open Problems. Definability for Very weak arithmetics

The second example is an infinite non-rooted non-oriented tree with all vertices of degree 3; $D_n(a, b)$ is true when the length of the shortest path between a and b is n. So, D_1 is the original relation.

It is easy to see, that we have the sublattice:



Fragment of the lattice for the infinite tree

Open problem 5

Is there anything else?

More open problems on definability

Decidability

Open problem 6

Elgot, Calvin C. and Rabin, Michael O.:
Decidability and Undecidability of Extensions of Second (First) Order Theory of (Generalized) Successor. J. Symb. Log., Vol. 31, No. 2, 169–181 (1966).

Does there exist a structure with maximally decidable theory?

Soprunov in 1988 proved (using forcing arguments) that that there is no maximal decidable space if we use weak monadic language for definability instead of our standard language.

Bès and Cégielski found a structure with a decidable theory (even monadic theory) such that any expansion of it by a constant has an undecidable theory.

Open problem 7 (generic)

Does an element of a space (given by formula) belong to a subspace generated by a given set of elements? — for different structures.

Positive and negative results on this for homogeneous structures: Bodirsky, Manuel, Pinsker, Michael and Tsankov (2011).

For $\langle \mathbb{Z}, \{+\} \rangle$ the positive answer was given by Semenov and Muchnik.

Muchnik's self-definability

Muchnik's Definition 1999–2003

S is *self-definable* iff it has a finite set of generators Σ and sequence $F_1, \dots, F_n, \dots$ such that for $n = 1, 2, \dots$

1. F_n is a closed formula in signature $\Sigma \cup \{P\}$, where P is an n-ary symbol.
2. F_n is true iff we take as interpretation of P an element from S.

Following Tarski.
Homogeneous spaces are self-definable.
The field of algebraic reals — not.

Muchnik's self-definability

Muchnik's Theorem

The space $\langle \mathbb{N}; + \rangle$ is self-definable.

Muchnik's Problem

Find more self-definable (and not-self-definable) structures.

Problems. Examples

- Elgot – Rabin 1966 Is there a maximal decidable (elementary) definability space.
- Albert Muchnik 1975, ref. to P.S. Novikov 1960-s — To describe the definability structure for $\langle \mathbb{Z}; + \rangle$.
- Thomas Conjecture (1991): The lattice of every homogeneous f.g. definability space is finite.
- Andrei Muchnik 2008 (1999) Find more self-definable spaces (for example — investigate $\langle \mathbb{Q}; +1 \rangle$).
- Semenov, Soprunov (2011) Is quantifier height invariant of choice of f.g.? Are their 'natural' examples of height 3, ...?