Why is it hard to separate $\mathsf{T}^2_2(\mathsf{R})$ from $\mathsf{T}^\infty_2(\mathsf{R})$ via forcing?

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The presentation is primarily based on the paper **Partially Definable Forcing and Bounded Arithmetic** by Albert Atserias and Moritz Müller [2].

Introduction

Setup

“We are interested in the following problem. Given a nonstandard model M arithmetic we want to expand it by interpreting a binary relation symbol R such that R^M does something prohibitive, e.g. violates the pigeonhole principle in the sense that R^M is a bijection from $n+1$ onto n for some (nonstandard) $n \in M$. The goal is to do so while preserving as much as possible from ordinary arithmetic. More precisely, we want the expansion (M, R^M) to model the least number principle for a class of formulas as large as possible.” [2][p. 1]

How (not) to solve the problem

- Consider the following R^M: any nonstandard number $m < n+1$ is mapped to $m-1$, while any standard m is mapped to itself.
- Such R^M obviously is a bijection from $n+1$ onto n.
- However, already the formula $R(x, x-1)$ violates the least number principle.

Theorems of Paris and Wilkie, and Riis

Theorem ([7])
It is possible to construct R^M *as above such that* (M, R^M) *preserves the least number principle for formulas with existential quantifiers.*

Theorem ([9])
It is possible to construct R^M *as above such that* (M, R^M) *preserves the least number principle for formulas with existential quantifiers and universal quantifiers bounded by* $b_0 < n^{o(1)}$*, i.e.* n *raised to an infinitesimal power.*

- In fact, both theorems work for a number of other true principles besides the mentioned pigeonhole principle.
- Both theorems are based on (slightly different) forcing arguments.

Theorem of Ajtai

Theorem ([1])
It is possible to construct R^M as above such that (M, R^M) preserves the least number principle for formulas with existential and universal quantifiers bounded by b_0 which is bigger than any standard power of n.

- The bounded was later extended to $b_0 < 2^{n^{o(1)}}$ ([5], [8]).
- Contrary to the two previous theorems, the fact that R^M must violate the pigeonhole principle is crucial here.

Bounded arithmetic

The previously mentioned theorems give the following corollaries

Corollary ([9])
The bounded arithmetic theory $\mathsf{T}^1_2(\mathsf{R})$ *does not prove the bijective pigeonhole principle.*

Corollary ([1])
The bounded arithmetic theory $\mathsf{T}^\infty_2(\mathsf{R})$ *does not prove the bijective pigeonhole principle.*

Forcing

Unifying approach

- The proofs for all three previously discussed theorems can be seen as instantiations of a general forcing argument.
- Following [2], it consists of choosing an appropriate poset of **conditions** $(P, \preceq)$ and specifying a **forcing relation** $\Vdash$ between the elements of P and sentences of the expanded language.
- Furthermore, it is enough to specify $\Vdash$ only for atomic sentences of the form $R(a, b)$ for $a, b \in M$ (due to conservativity of $\Vdash$).

Terminology

- We fix a countable model M (*ground model*) in a countable language L. Let L^* be a countable expansion of L. The **forcing langauge** $L^*(M)$ contains L^* and names for elements of M.
- **Forcing frame** $(P, \preceq)$ is a countable poset (which, in general, is not definable in M). Elements of P are called **conditions**.

Universal pre-forcing

- **Pre-forcing** $\Vdash$ is an arbitrary relation between conditions and $L^*(M)$-sentences. If $p \Vdash \varphi$, we say p **forces** φ.
- Pre-forcing $\Vdash$ is **universal** if it satisfies
  - $p \Vdash \neg\varphi$ iff $\forall q \preceq p : q \nVdash \varphi$;
  - $p \Vdash \varphi \wedge \psi$ iff $p \Vdash \varphi$ and $p \Vdash \psi$;
  - $p \Vdash \varphi \vee \psi$ iff $p \Vdash \neg(\neg\varphi \wedge \neg\psi)$;
  - $p \Vdash \forall x \chi(x)$ iff $\forall a \in M : p \Vdash \chi(a)$;
  - $p \Vdash \exists x \chi(x)$ iff $p \Vdash \neg\forall x \neg\chi(x)$.

Forcing

- A pre-forcing $\Vdash$ is called **forcing** if it satisfies
  - **Extension for atomic formulas**: if $p \Vdash \varphi$ and $q \preceq p$, then $q \Vdash \varphi$ (for atomic φ);
  - **Stability for atomic formulas**: if any $q \preceq p$ can be extended to $r \preceq q$ so that $r \Vdash \varphi$, then $p \Vdash \varphi$ (for atomic φ).

Lemma
Universal forcings satisfy stability for all sentences.

Lemma
For universal forcing $\Vdash$, $p \nVdash \varphi$ iff there exists $q \preceq p : q \Vdash \neg\varphi$.

Filters and dense sets

- A subset $G \subseteq P$ is called **filter** if it satisfies
  - $\forall p, q \in G\, \exists r \in G : r \preceq p, q$;
  - $\forall q \preceq p : q \in G \to p \in G$.
- A subset $D \subseteq P$ is called **dense** if it satisfies
  - $\forall p \in P\, \exists q \in D : q \preceq p$.

Generic filters

A filter G is called **generic** if it intersects *sufficiently many* dense subsets of P.

Generic associates

- Given a generic G, one wants to define the **generic associate** structure $M[G]$ satisfying $M[G] \vDash \varphi$ iff $\exists p \in G\ p \Vdash \varphi$.
- This can be done provided $\Vdash$ satisfies certain *definability conditions* [2, Definition 2.16], [2, Truth lemma 2.19].

Lemma (Forcing completeness)
Let $\Vdash$ be universal forcing and assume $M[G]$ is defined for all generic G. Then, $p \Vdash \varphi$ iff $M[G] \vDash \varphi$ for all generic G containing p.

Conservative forcing

- A forcing $\Vdash$ is called **conservative** iff, for every atomic $L(M)$-sentence φ, $p \Vdash \varphi$ iff $M \vDash \varphi$.
- In fact, for universal forcings so that $M[G]$ is defined for every generic G, the above characterizes when $M[G]$ is an expansion of M.

Partial definabiliy

- We fix a universal forcing $\Vdash$ and let L contain a binary $\leq$ with M interpreting $\leq$ as a total linear order so that all definable subsets contain $\leq$-smallest elements.
- We say $\Vdash$ is **definable** for an $L^*(M)$-formula $\varphi(\overline{x})$ if for every p the set $\{\overline{a} \mid p \| \varphi(\overline{a})\}$ is definable in M.

Theorem (Principal theorem)
Let Φ be a class of $L^*(M)$-formulas. Assume $\Vdash$ is definable for all formulas from Φ. Then, any generic expansion $M[G]$ satisfies the least number principle for Φ.

Lemma
The class of all formulas Ψ for which $\Vdash$ is definable is closed under disjunctions and existential quantification.

Theorem of Paris and Wilkie

Theorem ([7])
It is possible to construct R^M as above such that (M, R^M) preserves the least number principle for formulas with existential quantifiers.

- To prove the theorem we let $(P, \preceq)^{PW}$ be the poset of finite mathcings of $K_{n+1,n}$ ordered by inverse inclusion. We then define $p \Vdash_{PW} R(a, b)$ iff $\{a, b\} \in p$.
- It is then not that hard to show that $\Vdash_{PW}$ is definable for open formulas.

Theorem of Riis

Theorem ([9])
It is possible to construct R^M as above such that (M, R^M) preserves the least number principle for formulas with existential quantifiers and universal quantifiers bounded by $b_0 < n^{o(1)}$, i.e. n raised to an infinitesimal power.

- To prove the theorem we let $(P, \preceq)^{Ri}$ be the poset of codable mathcings of $K_{n+1,n}$ of sizes $\leq b_0^c$ ordered by inverse inclusion. We then define $p \Vdash_{Ri} R(a, b)$ iff $\{a, b\} \in p$.
- It is then not that hard to show that $\Vdash_{Ri}$ is definable for $\Delta_0^{b_0}(R)$-formulas.

Theorem of Ajtai

Theorem ([1])
It is possible to construct R^M as above such that (M, R^M) preserves the least number principle for formulas with existential and universal quantifiers bounded by b_0 which is bigger than any standard power of n.

- To prove the theorem we let $(P, \preceq)^{Aj}$ be the poset of codable mathcings of $K_{n+1,n}$ of sizes $\leq n - n^{\epsilon}$ for standard rational $\epsilon < 0$ ordered by inverse inclusion. We then define $p \Vdash_{Aj} R(a, b)$ iff $\{a, b\} \in p$.
- It is then highly non trivial to show that $\Vdash_{Aj}$ is definable for $\Delta_0^{b_0}(R)$-formulas (up to b_0).

Summary

- All three posts $(P, \preceq)^{PW}, (P, \preceq)^{Ri}, (P, \preceq)^{Aj}$ and forcing relations $\Vdash_{PW}, \Vdash_{Ri}, \Vdash_{Aj}$ are defined almost exactly the same.
- The only difference are the sizes of matchings of $K_{n+1,n}$ forming the underlying frames.

Limitations

Explicit formula violating LNP

Theorem
Let b_0 as in the Riis' theorem (i.e., $b_0 < n^{o(1)}$). Then, any generic expansion $M[G]$ violates the leas number principle for the formula $\varphi(x)$ defined as

$$\forall y \in [x, x + \delta(n - x)] : R(y, y),$$

where δ is standard rational $< 1/2$.

- In fact, it seems the argument works for conditions being as large as $n^{\epsilon/2}$ for an arbitrary standard rational $\epsilon < 1$.
- What about n^{ϵ} or ϵn (note that we cannot expect this to work for conditions of sizes $n - n^{\epsilon}$)?

Tournament principle

Definition ([4, 12.1])
A **tournament** is a directed graph (V, E) with exactly one directed edge between any two nodes.

A set $X \subseteq V$ is said to be **dominating** if, for any vertex $w \in V \setminus X$, there is a $v \in X$ so that $(v, w) \in E$.

- A well-know fact is that a tournament on m vertices contains a dominating set of size $\leq |m| + 1$ (see, e.g. [6, 2.5]).
- The tournament principle was shown to be provable in the bounded arithmetic theory $\mathrm{APC}_2(E)$ by E. Jeřábek in [3].

Violating tournament principle

Theorem
Assume $n + 1 = \binom{m}{2}$ *for a suitable m. Assume the sizes of the conditions are* $< m^{\epsilon}$ *for standard rational* $\epsilon < 1$*. Then, any generic expansion* $M[G]$ *violates the tournament principle for the graph* $([m], E)$*, defined as* $\min(a, b) E \max(a, b)$ *iff* R *maps the pigeon* $\{a, b\}$ *to an even hole.*

- Here, again, we see that conditions of size $< n^{\epsilon/2}$ are “too small” to force $T^2_2(R)$.
- Is it possible to have a “smarter” encoding of an orientation of a graph V using the relation R?

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