Ramsey-like theorems for separable permutations

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Joint work with Ludovic Patey

Introduction

Base theory RCA_0

Base theory (corresponding to computable mathematics) RCA_0:

- Robinson's arithmetic Q
- Δ^0_1-comprehension (The computable sets exists)
- Σ^0_1-induction (Every set of finite cardinality is bounded)

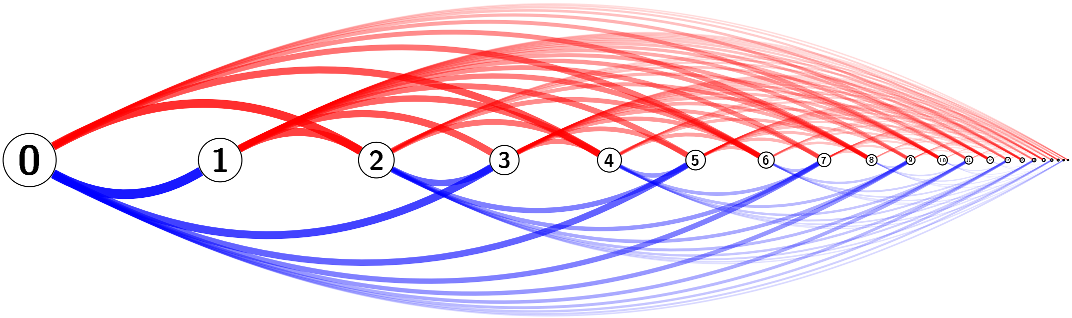
The "Big Five"

Modulo RCA_0, most theorems of ordinary mathematics are equivalent to one the following theories:

$$
\begin{array}{cl}
\Pi^1_1 - CA_0 & : \text{ impredicativism} \\
\downarrow & \\
ATR_0 & : \text{ transfinite recursion} \\
\downarrow & \\
ACA_0 & : \text{ second-order version of PA} \\
\downarrow & \\
WKL_0 & : \text{ compactness arguments} \\
\downarrow & \\
RCA_0 & : \text{ constructive mathematics}
\end{array}
$$

Ramsey's theorem for pairs

RT^2_2 is the statement that for any 2-coloring of the infinite clique $[\mathbb{N}]^2$:



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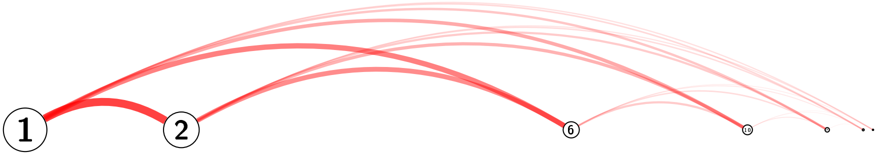


There exists an infinite monochromatic sub-clique.

$$\mathsf{RCA}_0 \nvdash \mathsf{RT}^2_2$$

$\mathsf{RCA}_0 + \mathsf{RT}^2_2 \vdash \mathsf{I}\Delta_2$
$\mathsf{RCA}_0 \nvdash \mathsf{I}\Delta_2$

There exists a computable instance of RT^2_2 with no computable solution

First-order part

Second-order part

Models

Models $\mathcal{M} = (M, S, 0, 1, \times, +, <)$:

- M is the first-order part.
- $S \subseteq \mathcal{P}(M)$ is the second-order part.
- $0, 1, \times, +, <$ are constants/functions/relations on M.

We will mostly consider *ω-models*, where M is the set of standard integers ω and $0, 1, \times, +, <$ have their canonical interpretation.

We say that a statement φ_1 implies φ_2 over ω-models, if every ω-models of φ_1 (plus RCA_0) is a model of φ_2.

$$
\begin{tikzcd}
\Pi^1_1-\mathsf{CA}_0 \arrow[d] & \\
\mathsf{ATR}_0 \arrow[d] & \\
\mathsf{ACA}_0 \arrow[d] \arrow[dr] & \\
\mathsf{WKL}_0 \arrow[d] & \mathsf{RT}^2_2 \arrow[dl] \\
\mathsf{RCA}_0 &
\end{tikzcd}
$$

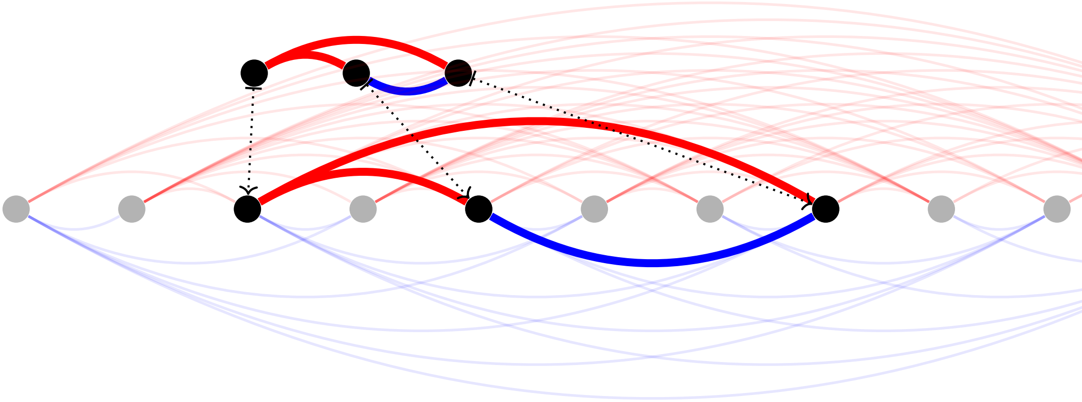
RT^2_2 escapes the structural phenomenon of the "Big Five".

What about weakenings (and strengthenings) of RT^2_2 ?

- Restricting to a certain class of colorings.
- Relaxing the monochromatic constraint.

Definition (pattern)

A *pattern* of size ℓ is a 2-coloring $p : [\ell]^2 \to 2$ of the clique of size ℓ.



A pattern p *appear* in $f : [\mathbb{N}]^2 \to 2$ if it can be embedded into it by an increasing function. Otherwise, f *avoids* p.

Definition

- Let $\mathsf{RT}^2_2(p)$ be the statement: "For every coloring $f : [\mathbb{N}]^2 \to 2$, there exists an infinite set H such that $f{\restriction}[H]^2$ avoids p"
- Let $\mathsf{coRT}^2_2(p)$ be the statement: "For every coloring $f : [\mathbb{N}]^2 \to 2$ avoiding p, there exists an infinite monochromatic subset"

Proposition

If p is monochromatic, then $\mathsf{RT}^2_2(p)$ is a false statement, otherwise $\mathsf{RT}^2_2(p)$ is true. (We will only consider non-monochromatic patterns from now on).

Proof.

- Let p be monochromatic for some color $c < 2$, and let $f : [\mathbb{N}]^2 \to 2$ be constant equal to c, then no subset of $\mathbb{N}$ can f-avoid p. Hence, $\mathsf{RT}^2_2(p)$ doesn't hold.
- Let p be non-monochromatic and $f : [\mathbb{N}]^2 \to 2$ be any coloring. By RT^2_2 let H be infinite and f-monochromatic, $f{\restriction}[H]^2$ avoids p, thus $\mathsf{RCA}_0 + \mathsf{RT}^2_2 \vdash \mathsf{RT}^2_2(p)$.

$\square$

Proposition

$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \leftrightarrow \mathsf{RT}^2_2(p) \wedge \mathsf{coRT}^2_2(p)$

Proposition (Mimouni, Patey (2025))

For every pattern p, $\mathsf{RT}^2_2(p)$ is not computably true.

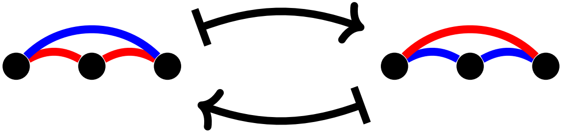
In fact, they proved a stronger property and showed that $\mathsf{RT}^2_2(p)$ implies the existence of a $\emptyset'$-DNC function, i.e. a total function $g : \mathbb{N} \to \mathbb{N}$ such that $g(e) \neq \Phi^{\emptyset'}_e(e)$ for every $e \in \mathbb{N}$.

Equivalently, we have a function $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $h(k, e)$ doesn't satisfy $\phi_e(x)$ for ϕ_e the e-th Σ_2-formula if $|\{x : \phi_e(x)\}| \leq k$.

Dual pattern

Definition

The *dual* $\bar{p}$ of a coloring/pattern p is the coloring/pattern whose colors are inverted.



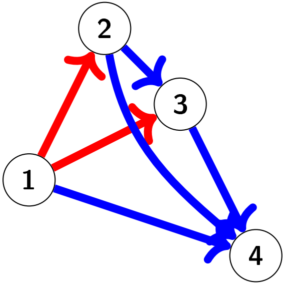
Proposition

$\mathsf{RCA}_0 \vdash (\forall p)\mathsf{RT}^2_2(p) \leftrightarrow \mathsf{RT}^2_2(\bar{p})$

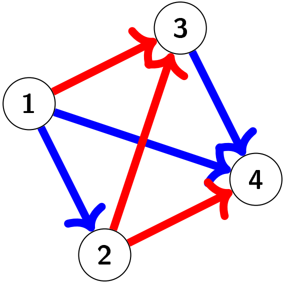
Proof.

Assume $\mathsf{RT}^2_2(p)$ and let $f : [\mathbb{N}]^2 \to 2$ be a coloring. Let $\bar{f}$ be the dual coloring of f and let $H \subseteq \mathbb{N}$ be infinite and such that $\bar{f}{\restriction}[H]^2$ avoids p, then $f{\restriction}[H]^2$ avoids $\bar{p}$. $\square$

Colorings avoiding  and  can be seen as permutations:



Permutation 2431



Permutation 3142

Definition (EM)

EM is the statement: for every coloring $f : [\mathbb{N}]^2 \to 2$, there exists an infinite set H such that $f \restriction [H]^2$ is transitive.

Definition (ADS)

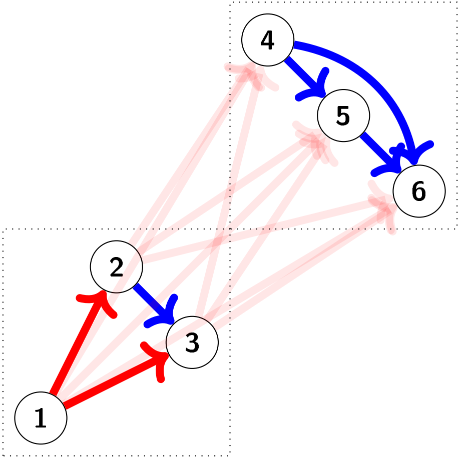
ADS is the statement: for every linear order $(\mathcal{L}, <_{\mathcal{L}})$, there exists an infinite monotonous sequence.

- $\mathsf{RCA}_0 \vdash \mathsf{EM} + \mathsf{ADS} \leftrightarrow \mathsf{RT}^2_2$ (Bovykin/Weiermann/Montálban)
- $\mathsf{RCA}_0 + \mathsf{EM} \nvdash \mathsf{RT}^2_2$ (Lerman/Solomon/Towsner)
- For every non-transitive pattern p : $\mathsf{RCA}_0 \vdash \mathsf{EM} \to \mathsf{RT}^2_2(p)$, hence $\mathsf{RCA}_0 + \mathsf{RT}^2_2(p) \nvdash \mathsf{RT}^2_2$.

Separable permutations

Definition (Bose, Buss & Lubiw (1998))

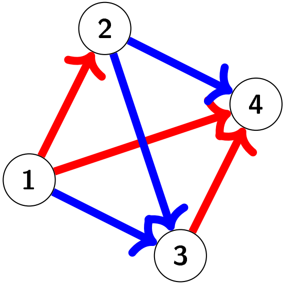
A *separable permutation* is one that can be obtained from the trivial permutation on 1 (or 0) elements by a series of direct ($\oplus$) and skew ($\ominus$) sums.



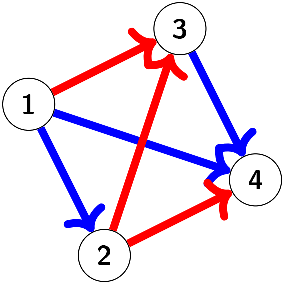
$$\underline{132 \oplus 321 = 132654}$$

Proposition (Bose, Buss & Lubiw (1998))

The separable permutations are exactly those avoiding the patterns 2413 *and its dual* 3142



Permutation 2413



Permutation 3142

The number of separable permutations when n grows is:
$1, 2, 6, 22, 90, 394, 1806, 8558, \ldots$. (Schröder numbers, sequence A006318 in the OEIS)

Theorem (Mimouni, Patey (2025))

Let p be a non-separable permutation, then $\mathsf{RCA}_0 + \mathsf{RT}^2_2(p) \nvdash \mathsf{RT}^2_2$

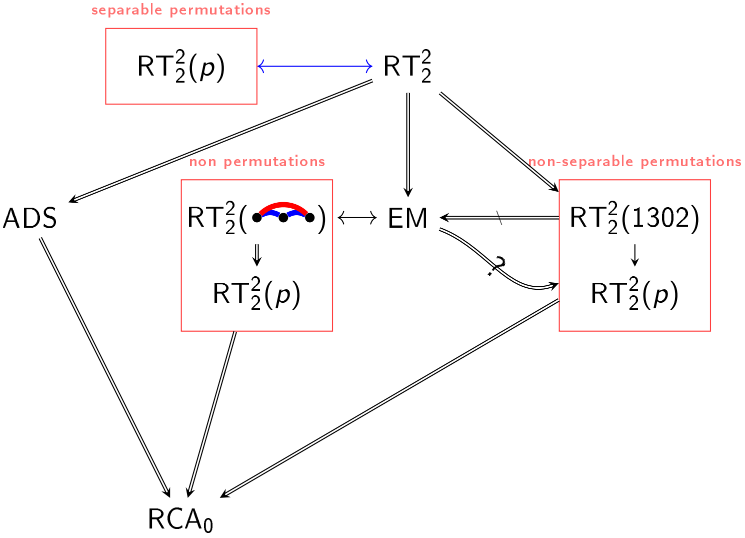
Theorem (L., Patey (2025))

$\mathsf{RT}^2_2(p)$ *implies* RT^2_2 *over ω-models if and only if p is a separable permutation.*

Proof sketch.

- Assuming that $\mathsf{RT}^2_2(p)$ holds in an ω-model, we want to prove that $co\mathsf{RT}^2_2(p)$ (and thus RT^2_2) also holds.
- Let $f : [\mathbb{N}]^2 \to 2$ be a coloring avoiding p.
- By $\mathsf{RT}^2_2(p)$, there exists a f'-DNC function g.
- There exists a probabilistic algorithm computing an infinite f-monochromatic set, and g computes such a set that will therefore be in the model.

$\square$



Implications over ω-models of RCA_0

First-order part

Proposition (L., Patey (2025))

Let p be a permutation, then $\mathsf{RCA}_0 + \mathsf{RT}^2_2(p) \vdash \mathsf{I}\Delta_2$

Question

Does $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2(p) \to \mathsf{RT}^2_2$ for every separable permutation p ?

Question

What is the first-order part of $\mathsf{RT}^2_2(p)$, for any pattern p ?

References

Ahmed Mimouni and Ludovic Patey.
Ramsey-like theorems and immunities, 2025.

Prosenjit Bose, Jonathan F. Buss, and Anna Lubiw.
Pattern matching for permutations.
Inform. Process. Lett., 65(5):277–283, 1998.