Gauging the strength of (infinite) Ramsey's theorem for pairs

(first-order consequences, the effect of the base theory, influence on proof size)

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This talk

A survey-ish talk on the study of the first-order consequences of infinite Ramsey's Theorem for pairs and two colours, i.e. (roughly) what statements in the language of 1^{st} order arithmetic follow from Ramsey's Theorem over a suitable base theory.

Emphasis on themes that are of particular interest to the speaker and (hopefully) part of the JAF audience:

- the effect of weakening the base theory,
- the effect of Ramsey's Theorem on proof size.

Reverse mathematics: usual setup

- Reverse mathematics studies the strength of axioms needed to prove mathematical theorems. Most commonly, this is done by deriving implications between the theorems and/or set existence principles expressed in second-order arithmetic.
- The implications are proved in a relatively weak base theory, corresponding in some sense to computable mathematics.
- Often, the theorem is Π^1_2 of the form $\forall X\,\exists Y\psi$, and its strength is related to the difficulty of computing Y given X.

Language of second-order arithmetic

L_2 has two sorts of variables:

- first-order sort $x, y, z, \ldots, i, j, k \ldots$ for natural numbers,
- second-order sort $X, Y, Z, \ldots$ for subsets of $\mathbb{N}$,
- extra-logical symbols: $+, \cdot, \leq, 0, 1\,; x \in X$.

Σ^0_n: class of formulas with n first-order quantifier blocks, beginning with $\exists$, then only bounded quantifiers $\exists x \leq t$, $\forall x \leq t$.
Π^0_n: dual class, beginning with $\forall$.
We drop the ^{0} when set parameters are not allowed (Σ_n, Π_n).

Example: each r.e. subset of $\mathbb{N}$ can be defined by a Σ_1 formula.
$P \neq NP$ is a Π_2 statement. So is the twin prime conjecture.
Stating "the set X is infinite" is Π^0_2 (not Π_2, we can say it is $\Pi_2(X)$).

Σ^1_n, Π^1_n: defined using alternations of second-order quantifiers.

Fragments of second-order arithmetic

ACA_0 ("arithmetical comprehension"):

- Comprehension: $\exists X\, \forall k\, (k \in X \Leftrightarrow \varphi(k))$ for φ arithmetical (with no second-order quantifiers).
- Induction: $\forall X\, (0 \in X \wedge \forall k\, (k \in X \Rightarrow k+1 \in X) \Rightarrow \forall k\, (k \in X))$.
- Conservative over PA.
- Proves e.g.: all of freshman analysis, Ramsey's Thm (for pairs, triples etc.), König's Lemma, completeness thm for FO logic...

RCA_0 ("recursive comprehension", usual base theory):

- Only Δ^0_1 comprehension ($\nvdash$ existence of undecidable sets).
- For technical reasons induction strengthened to $\mathrm{I}\Sigma^0_1$.
- Formalizes e.g. all primitive recursive constructions.
- Conservative over $\mathrm{I}\Sigma_1$.

Reverse mathematics: the traditional picture

Most theorems analyzed, especially early on (1970's, 80's), turn out to be either provable in RCA_0 ($\approx$ "computably true") or equivalent over RCA_0 to one of four increasingly strong theories:

$$\mathsf{WKL}_0, \quad \mathsf{ACA}_0, \quad \mathsf{ATR}_0, \quad \Pi^1_1\text{-}\mathsf{CA}_0.$$

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WKL_0 is $\mathsf{RCA}_0 + \mathsf{WKL}$, where WKL is Weak König's Lemma: each infinite tree in $2^{<\mathbb{N}}$ has infinite path.

- equivalent to e.g. compactness or completeness for FO logic,
- strictly stronger than RCA_0: fails in the computable sets,
- strictly weaker than ACA_0:
  in fact, arithmetically (and even Π^1_1-) conservative over RCA_0.

Weird case: Ramsey's Theorem

RT^n_2 is "for each $f : [\mathbb{N}]^n \to 2$, there is infinite H homogeneous for f" (i.e. f is constant on $[H]^n$).

Over RCA_0, each of $\mathsf{RT}^3_2, \mathsf{RT}^4_2, \ldots$ is equivalent to ACA_0.

But RT^2_2 is weird:

- does not follow from WKL_0 (implicit in Jockusch 1972),
- does not imply ACA_0 (Seetapun-Slaman 1995),
- in fact, does not even imply WKL_0 (Liu 2012).

Example: arithmetical comprehension from RT^3_2

We use RT^3_2 to prove that for any set S,
the set S' of machines that halt with oracle S also exists.

Consider the 2-colouring of triples $x < y < z$:

$$f(x, y, z) = \begin{cases} 1 & \text{if there is a Turing machine with code at most } x \\ & \text{that halts with oracle } S \text{ before } z \text{ but not before } y, \\ 0 & \text{otherwise.} \end{cases}$$

By RT^3_2, there is an infinite homogeneous set H for f.
If H has colour 1, then it has at most $\min H + 3$ elements.
So, H must have colour 0 (this uses $I\Sigma^0_1$!).

So: with oracle S, machine e halts iff it halts before the second smallest element of H after e (a $\Delta_1(H, S)$ property, so corresponds to a set).

First-order parts

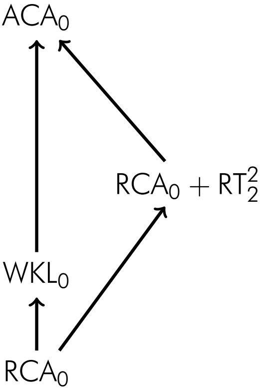
It is interesting to ask not only what second-order statements, but also what first-order statements (sentences about the natural numbers) a given theory implies.

We will move rather freely between two related objects:

- the first-order part of a theory (the arithmetical sentences it proves),
- the Π^1_1 part (the Π^1_1 sentences/arithmetical formulas it proves).

E.g. the first-order part of RCA_0 is $\mathrm{I}\Sigma_1$, the Π^1_1 part is $\mathrm{I}\Sigma^0_1$.

Π^1_1 parts of some theories



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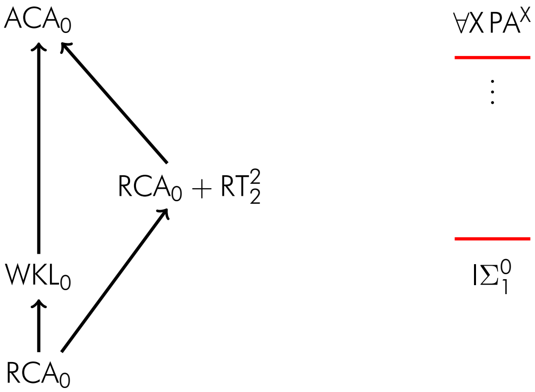

Question:

What is the first-order/Π^1_1 part of $RCA_0 + RT^2_2$?

Π^1_1 part of RT^2_2: first observation

Recall: $B\Sigma^0_1 \subseteq I\Sigma^0_1 \subseteq B\Sigma^0_2 \subseteq I\Sigma^0_2 \ldots$, where $B\Sigma^0_n$ is Σ^0_n collection:

$$\forall x \leq v\, \exists y\, \psi(x,y) \Rightarrow \exists w\, \forall x \leq v\, \exists y \leq w\, \psi(x,y).$$

Theorem (Hirst 1987)

$\mathsf{RCA}_0 + \mathsf{RT}^2_2 \vdash B\Sigma^0_2$.

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Theorem (Hirst 1987)

$RCA_0 + RT^2_2 \vdash B\Sigma^0_2$.

Proof: Consider $B\Sigma^0_2$ in the equivalent form of the infinite PHP: "if $f : \mathbb{N} \to k$ for some k, then $f^{-1}[\{\ell\}]$ is infinite for some $\ell < k$".

$$g(x, y) = \begin{cases} 1 & \text{if } f(x) \neq f(y), \\ 0 & \text{if } f(x) = f(y). \end{cases}$$

By RT^2_2, there is an infinite homogeneous set H for g.
Once again, H must have colour 0 (and again, this uses $I\Sigma^0_1$!). □

State of the art, around 2018

$\mathsf{RCA}_0 + \mathsf{RT}^2_2$:

(a) is Π^1_1-conservative over $\mathsf{I}\Sigma^0_2$. [Cholak-Jockusch-Slaman 2001]

(b) does not imply $\mathsf{I}\Sigma^0_2$. [Chong-Slaman-Yang 2017]

(Implicitly: is Π^0_4-conservative over $\mathsf{B}\Sigma^0_2 + \{\mathsf{WO}(\omega^\omega), \mathsf{WO}(\omega^{\omega^\omega}), \ldots\}$)

(c) is Π^0_3-conservative over $\mathsf{B}\Sigma^0_2$ and $\mathsf{I}\Sigma^0_1$. [Patey-Yokoyama 2018]

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Concerning CJS and CSY, just two notes:

- For suitable ctble $(M, \mathcal{X})$ and $\mathcal{X} \ni f : [M]^2 \to 2$, one finds homogeneous $H \subseteq M$ s.t. $(M, \mathcal{X} \cup \{H\})$ still "good enough".
- One needs to split the construction of H into parts corresponding to a decomposition of RT^2_2, here into stable Ramsey's Theorem SRT^2_2 and the cohesive set principle COH.

Patey-Yokoyama

Theorem (Patey-Yokoyama 2018)

$\mathsf{RCA}_0 + \mathsf{RT}^2_2$ is Π^0_3-conservative over $\mathsf{I}\Sigma^0_1$.

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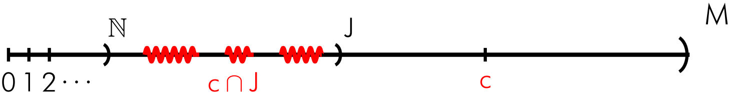
The required model of RT^2_2 can be built on a cut in a nonstandard model of $\mathrm{I}\Sigma^0_1$, with the coded sets as second-order universe.



Main combinatorial engine: for every $k \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $\mathrm{RCA}_0 \vdash \left[\omega^m\text{-large} \to (\omega^k\text{-large})^2_2\right]$.

- $X \subseteq \mathbb{N}$ is ω^0-large iff it is nonempty,
- X is ω^{n+1}-large iff can be split into $\geq \min X$ successive ω^n-large pieces.

Proof size over RCA_0

Question (Patey-Yokoyama)

Does $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ have significant proof speedup over $\mathrm{I}\Sigma^0_1$ for Π^0_3 statements?

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Theorem (K-Wong-Yokoyama 2024)

No. There is a polynomial proof transformation.

Proof idea: Conceptually, the idea is to recast P-Y as a forcing interpretation: we do not define the cut satisfying RT^2_2 as such, but we define a forcing notion s.t. the generic object is such a cut.

Combinatorially, the core is to improve $\omega^{\mathsf{m}(\mathsf{k})}$-large $\to (\omega^{\mathsf{k}}\text{-large})^2_2$ to $\omega^{300\mathsf{k}}$-large $\to (\omega^{\mathsf{k}}\text{-large})^2_2$. This is pure finite combinatorics.

A surprise: proof size over RCA_0^*

RCA_0^* is a weaker theory sometimes considered instead of RCA_0:

- $\mathsf{I}\Sigma_1^0$ is removed, so only $\mathsf{I}\Delta_1^0$ is available.
- The axiom exp ($x \mapsto 2^x$ is total) is explicitly added.
- The Π_1^1 part is $\mathsf{B}\Sigma_1^0 + \mathrm{exp}$.

$\mathsf{RCA}_0^* + \mathsf{RT}_2^2$ is again Π_3^0-conservative over RCA_0^*. However:

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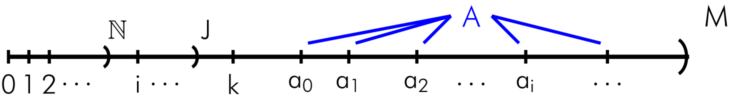
Theorem (K-Wong-Yokoyama 2024)

$\mathrm{RCA}_0^* + \mathrm{RT}^2_2$ has iterated exponential proof speedup over RCA_0^*, even w.r.t. proofs of $\Delta_0(\mathrm{exp})$ sentences.

Proof speedup over RCA_0^*: the idea

A new phenomenon in RCA_0^* is that an unbounded set A of natural numbers might not have a k-element finite subset for some k:



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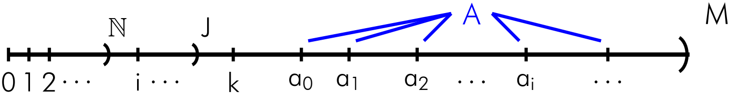

- If each unbounded set has at least i elts, cut-free consistency for RCA_0^* holds up to i.
- We can use $2^{k/2} \not\to (k)^2_2$ to show that RT^2_2 implies that this cut of those i is closed under exp. As a result, $\widetilde{Con}(RCA_0^*)$ holds on a smaller definable cut.
- By standard arguments, this gives proof speedup (on finite consistency statements).

Restricting this to computable unbounded sets, one can prove:

Theorem (K-Kowalik-Yokoyama 2023)
$RCA_0^* + RT^2_2$ is not Π_4-conservative over RCA_0^*.

WKL_0^* isomorphism theorem

WKL_0^* is $\mathsf{RCA}_0^* + \mathsf{WKL}$.

Theorem (Fiori Carones-K-Yokoyama-Wong 202X)

Let $(M, \mathcal{X})$, $(M, \mathcal{Y})$ be countable models of WKL_0^* s.t. $(M, \mathcal{X} \cap \mathcal{Y}) \models \neg \mathrm{I}\Sigma_1^0$. Then $(M, \mathcal{X})$ and $(M, \mathcal{Y})$ are isomorphic.

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This gives a kind of second-order q.e. for $\mathsf{WKL}_0^* + \neg \mathsf{I}\Sigma_1^0$.

It also means that if a Π_2^1-statement Ψ is Π_1^1-conservative over RCA_0^*, then it must hold that $\mathsf{WKL}_0^* + \neg \mathsf{I}\Sigma_1^0 \vdash \Psi$.

Does this say anything about conservativity over $\mathsf{RCA}_0 + \mathsf{B}\Sigma_2^0$?

Weak isomorphism theorem for $n = 2$

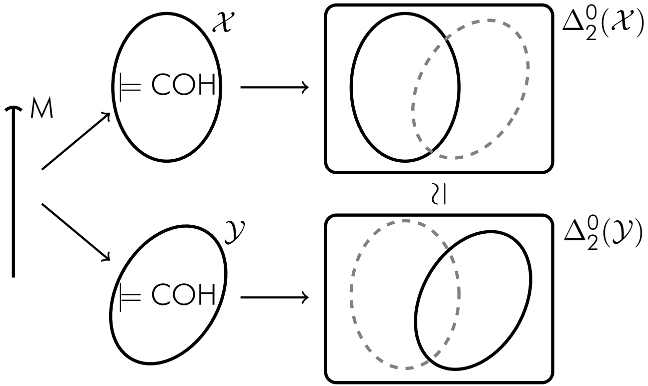
We need a principle saying "for every X and every $\Delta_2(X)$-definable infinite 0-1 tree, there is Y and a $\Delta_2(Y)$-definable infinite path".

That is known to be the weakening COH of RT^2_2 mentioned earlier!

Corollary

Let $(M, \mathcal{X}), (M, \mathcal{Y})$ be ctble models of $RCA_0 + B\Sigma^0_2 + COH$. If $(M, \mathcal{X} \cap \mathcal{Y}) \models \neg I\Sigma^0_2$, then $(M, \Delta^0_2\text{-Def}(M, \mathcal{X})) \simeq (M, \Delta^0_2\text{-Def}(M, \mathcal{Y}))$.

Weak isomorphism and conservativity: picture



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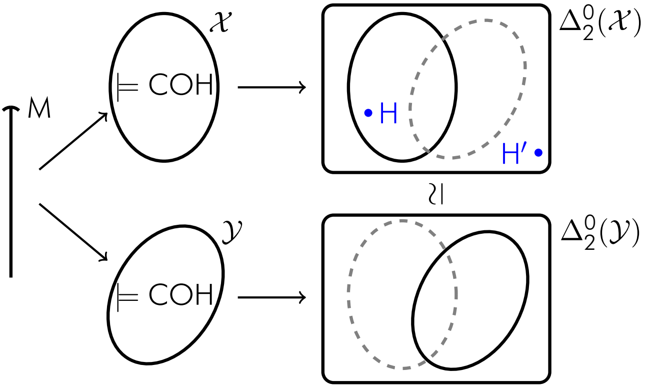

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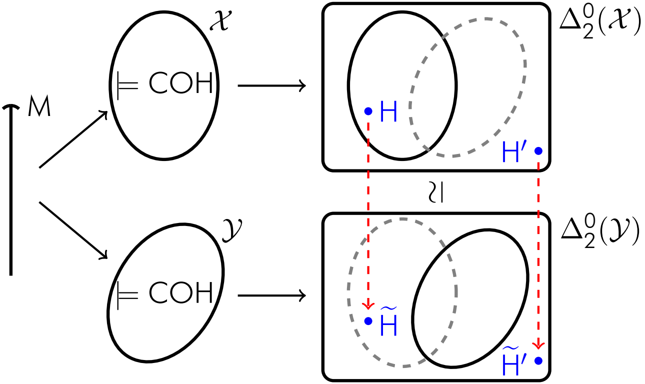

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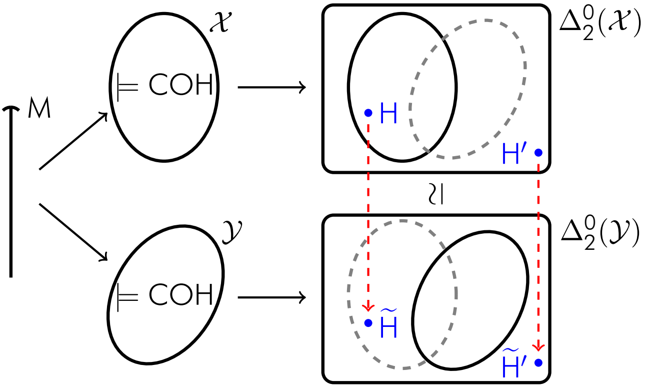

Moreover, a "sufficiently arbitrary" $(M, \mathcal{X}) \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ can be extended to $(M, \mathcal{Y})$ that additionally satisfies COH.

Weak isomorphism and conservativity: upshot

So, $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ proves:

"if $\mathsf{I}\Sigma^0_2$ fails and C is a solution to a suitable instance of COH, then for every $f\colon [\mathbb{N}]^2 \to 2$ there is a homogeneous $\Delta_2(C)$-definable class $\widetilde{H}$ that is close enough to a set". [roughly: low]

This statement is Π^0_5 (with an implicit $\forall$ set quantifier in front).

And if $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ proves it, then every "sufficiently arbitrary" model of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ can be extended so as to satisfy RT^2_2.

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Corollary (FKWY 202X)

FAE:

(i) $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ is Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.

(ii) $\mathsf{RCA}_0 + \mathsf{RT}^2_2$ is Π^0_5-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.

Recent work (1): two theorems by Le Houérou et al.

Le Houérou et al. have managed to strengthen the results of Chong-Slaman-Yang and Patey-Yokoyama, respectively, by proving:

Theorem (Le Houérou, Patey, Yokoyama 2025)
RT^2_2 is Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \{\mathsf{WO}(\omega^\omega, \omega^{\omega^\omega}, \ldots)\}$.

Theorem (Le Houérou, Patey, Yokoyama 202Y)
RT^2_2 is Π^0_4-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$.

Recent work (2)

Theorem (Ikari-K-Yokoyama 20??)

If Ψ is Π^1_2, then Π^1_1-conservativity of Ψ over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_n + \neg\mathsf{I}\Sigma^0_n$ always comes with a polynomial proof transformation.

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Proposition

$\neg\mathsf{WO}(\omega^\omega)$ is Π^1_1-conservative over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \neg\mathsf{I}\Sigma^0_2$.

References

1) Cholak, Jockusch, Slaman, On the strength of Ramsey's theorem for pairs, JSL 66 (2001), 1–55.
2) Chong, Slaman, Yang, The inductive strength of Ramsey's theorem for pairs, Adv. Math. 308 (2017), 121–141.
3) Patey, Yokoyama, The proof-theoretic strength of Ramsey's theorem for pairs and two colors, Adv. Math. 330 (2018), 1034–1070.
4) K, Wong, Yokoyama, Ramsey's theorem for pairs, collection, and proof size, J. Math. Log. 24 (2024), paper no. 2350007.
5) K, Kowalik, Yokoyama, How strong is Ramsey's theorem if infinity can be weak?, J. Symb. Log. 88 (2023), 620–639.
6) Fiori Carones, K, Wong, Yokoyama, An isomorphism theorem for models of Weak König's Lemma without primitive recursion, to appear in J. EMS, arXiv:2112.10876.
7) Le Houérou, Levy Patey, Yokoyama, Conservation of Ramsey's theorem for pairs and well-foundedness, Trans. AMS 378 (2025), 2157–2186.
8) Le Houérou, Levy Patey, Yokoyama, Π^0_4 conservation of Ramsey's theorem for pairs, arXiv:2404.18974.