

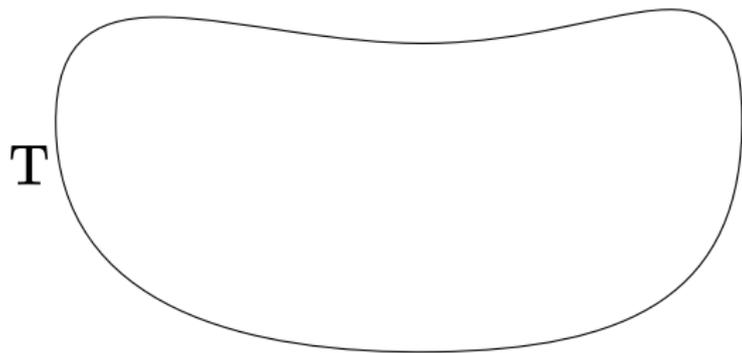
TIGHTNESS AND SEMANTIC TIGHTNESS
IN SUBTHEORIES OF PEANO ARITHMETIC

Piotr Gruza

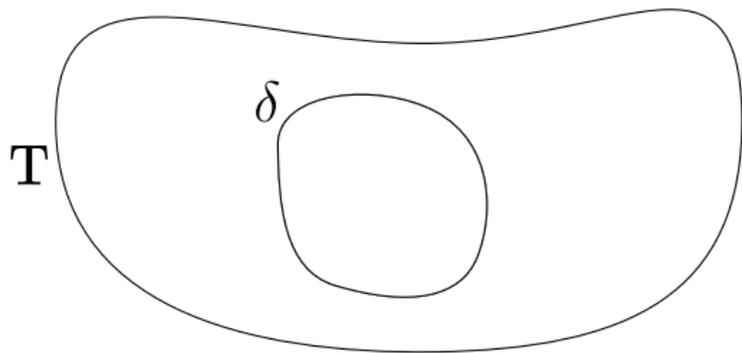
University of Warsaw

INTERPRETATIONS

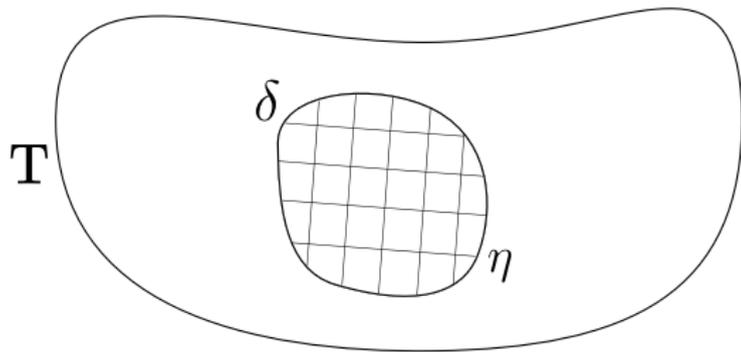
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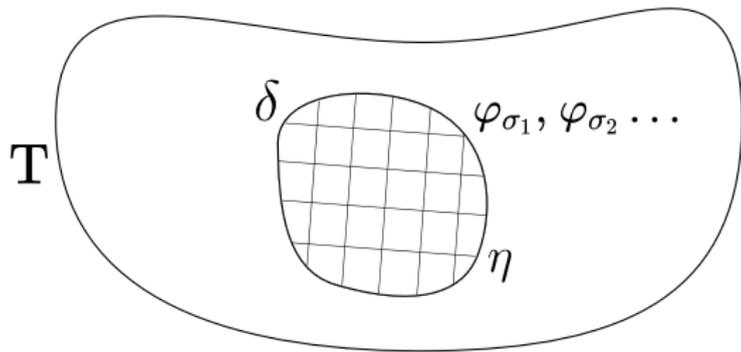
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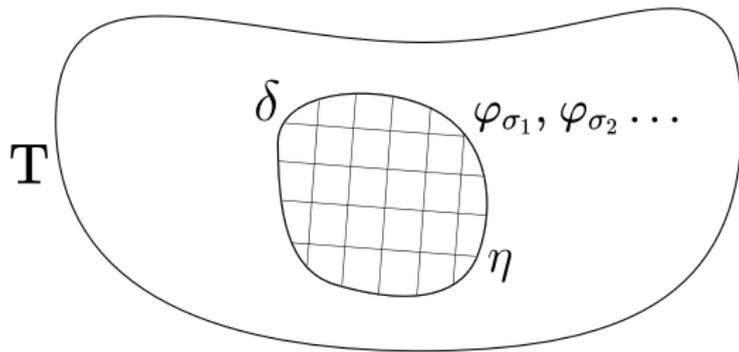
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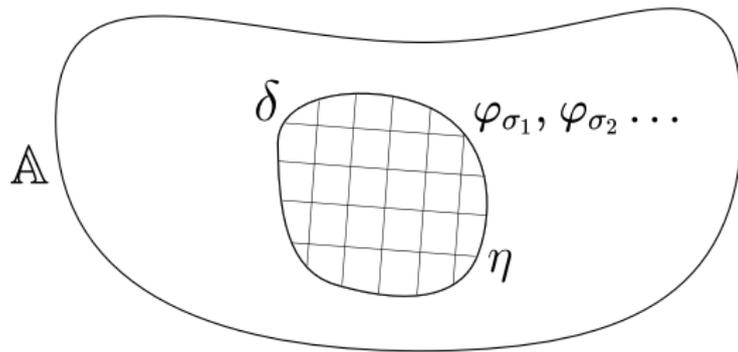


INTERPRETATIONS



$T \triangleright S$ if T proves the translations of all axioms of S .

INTERPRETATIONS



$A \triangleright B$ if the quotient structure is isomorphic to B .

BI-INTERPRETATIONS

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T and S are bi-interpretable if there exist $s: T \triangleright S$ and $t: S \triangleright T$ such that $t \circ s \simeq \text{id}_T$ and $s \circ t \simeq \text{id}_S$.

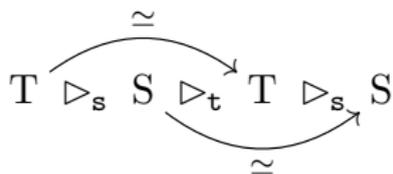
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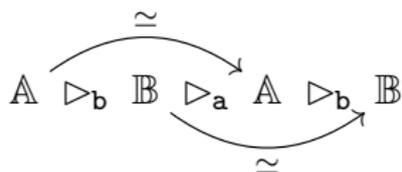
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$b_1, b_2: \mathbb{A} \triangleright \mathbb{B}$ are isomorphic ($b_1 \simeq b_2$) if there exists a formula of $\mathcal{L}_{\mathbb{A}}$ that defines in \mathbb{A} an isomorphism between the copies of \mathbb{B} given by b_1 and b_2 .

DEFINITION

\mathbb{A} and \mathbb{B} are bi-interpretable if there exist $b: \mathbb{A} \triangleright \mathbb{B}$ and $a: \mathbb{B} \triangleright \mathbb{A}$ such that $a \circ b \simeq \text{id}_{\mathbb{B}}$ and $b \circ a \simeq \text{id}_{\mathbb{A}}$.



BI-INTERPRETATIONS

THEOREM (KAYE & WONG)

The following theories are bi-interpretable:

- PA,
- $\text{ZF} \setminus \{\text{Inf}\} + \neg\text{Inf} + \text{TC}$.

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THEOREM (PETTIGREW)

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will be a semantically tight proper subtheory of PA .

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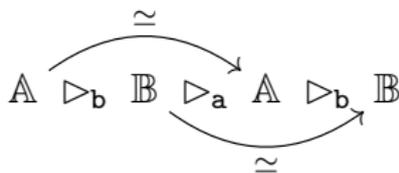
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Does there exist a semantically tight finite sequential theory?

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SUMMARY

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