

- PATRYK SZLUFIK, *Scott ranks of models of elementary arithmetic.*

This is joint work with Leszek Kołodziejczyk and Mateusz Lętyk.

It is a well-known result of Scott that every countable structure M can be characterized, up to an isomorphism within the class of all countable structures, by a sentence of $\mathcal{L}_{\omega_1\omega}$ – a logic which allows for countable conjunctions and disjunctions – in the signature of M . We will call such a sentence a Scott sentence of M . Formulae in $\mathcal{L}_{\omega_1,\omega}$ admit a hierarchy similar to the arithmetical one, starting with $\Sigma_0^{in} = \Pi_0^{in}$ as class of quantifier-free formulae. We then say that formula is Σ_α^{in} (resp. Π_α^{in}) iff it is of form $\bigvee_{i \in I} \exists \bar{x}_i \varphi_i(\bar{x}_i, \bar{y})$ (resp. $\bigwedge_{i \in I} \forall \bar{x}_i \varphi_i(\bar{x}_i, \bar{y})$) where each φ_i is Π_β^{in} (resp. Σ_β^{in}) for some $\beta < \alpha$. We then define the Scott rank of a model M as the least ordinal α such that M is uniquely characterized by a $\Pi_{\alpha+1}^{in}$ sentence. Equivalently, by the result [2] of Montalbán, we can define it as the least ordinal α such that every automorphic orbit of a tuple of elements of M is definable with Σ_α^{in} formula. Additionally, for a theory T , one can consider its Scott spectrum, i.e. the set of ordinals α such that some $M \models T$ has Scott rank α .

In recent work [3] Montalbán and Rossegger initiated the Scott analysis of models of Peano Arithmetic. They characterized the Scott spectra of completions of PA , showing in particular that the only $M \models PA$ of finite Scott rank is the standard model.

We investigate how this picture changes in the context of theories extending elementary arithmetic $I\Delta_0 + \exp$ (EA) that fail to satisfy the full induction scheme. In particular, we give lower bounds for Scott spectra of $I\Sigma_n + \neg B\Sigma_{n+1}$ and $B\Sigma_n + \neg I\Sigma_n$ and determine which infinite ordinals fall into these. We also give some general bounds on Scott ranks of models of EA , in particular we show that:

- (A) For every $n \geq 1$ every model of $B\Sigma_n$ has Scott rank at least $n + 1$
- (B) For each n there are models of $B\Sigma_n + \neg I\Sigma_{n+1}$ and $I\Sigma_n + \neg B\Sigma_{n+1}$ of Scott rank $n + 1$,
- (C) for every ordinal $\alpha \geq 1$ and for every n there is a model of $I\Sigma_n + \exp + \neg B\Sigma_{n+1}$ of Scott rank $n + 2 + \alpha$, in particular:

$$\{n + 1\} \cup [n + 3, \omega_1) \subseteq SS\text{p}(I\Sigma_n + \neg B\Sigma_{n+1}) \subseteq [n + 1, \omega_1)$$

- (D) for every ordinal $\alpha \geq 1$ and for every n there is a model of $B\Sigma_n + \exp + \neg I\Sigma_n$ of Scott rank $n + \alpha$ or $n + 1 + \alpha$, in particular:
 - (D.1) $\omega_1 \setminus \omega \subseteq SS\text{p}(B\Sigma_n + \neg I\Sigma_n) \subseteq [n + 1, \omega)$
 - (D.2) $SS\text{p}(B\Sigma_n + \neg I\Sigma_n) \cap \omega$ is cofinal in ω .
- (E1) for every n , every Σ_n -recursively saturated model of EA has the Scott rank at least n .
- (E2) each recursively saturated model of EA has Scott rank exactly $\omega + 1$.

Notably, the machinery used in the proofs of these facts doesn't resemble the machinery used in [3]. In contrast, our methods rely on general arithmetical arguments coupled with crucial property of EA , such that under Σ_n recursive saturation, Σ_n^{in} -types of elements are determined by their Σ_n -types and vice-versa. This „expressivity collapse” allows us to translate infinitary problem to purely arithmetical ones and solve them with available tools.

[1] RACHEL ALVIR, NOAM GREENBERG, MATTHEW HARRISON-TRAINOR & DAN TURETSKY, *Scott complexity of countable structures*, **Journal of Symbolic Logic**, vol. 86 (2021), no. 4, pp. 1706–1720.

[2] ANTONIO MONTALBÁN, *A robust Scott rank*, **Proceedings of American Mathematical Society**, vol. 143 (2015), no. 12, pp. 5427–5436.

[3] ANTONIO MONTALBÁN & DINO ROSSEGGER, *The Structural Complexity of Models of Arithmetic*, **Journal of Symbolic Logic**, vol. 89 (2024), no. 4, pp. 1703–1719.