DEFINABILITY LATTICES FOR VERY WEAK ARITHMETICS. OPEN PROBLEMS AND RECENT ADVANCES

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Abstract. We study definability (reduct) latticies for some descrete numerical and graph structures.

Keywords: definability, reducts, definability lattice, automorphisms, Svenonius theorem.

1. Introduction. Definitions and History

1.1. **Origins.** The origins of definability theory can be seen in papers of Italian mathematicians of the 19th century, which led, in particular, to the idea of the automorphism method by A. Padoa. Later, the importance of definability theory was repeatedly emphasized by A. Tarski. The theorems of Gödel and Tarski, in essence, asserted the definability of proof and the non-definability of truth in finitist mathematics. The more detailed historical survey can be found in [SSU14].

Definability spaces, in alternative terms – reducts can be naturally defined in a 'coordinate-free' way (see [SS21]). For a given space (i.d. of a structure) its subspaces form the definability lattice of the structure.

A famous Russian logician Petr Novikov proposed to study the definability lattice of Presburger arithmetic to his pupil Albert Muchnik, the last – to his pupil Alexei Semenov. The study is far from complete until now.

Further on, we consider only countable structures.

1.2. **Huntington's Relations.** E. Huntington at the beginning of the 20th century considered three relations defined through every linear order:

$$B(a,b,c) \rightleftharpoons (a<b<c) \vee (a>b>c) \text{ } \textit{Between}$$
$$C(a,b,c) \rightleftharpoons (a<b<c) \vee (b<c<a) \vee (c<a<b) \text{ } \textit{Cycle}$$
$$S(a,b,c,d) \rightleftharpoons \Big(B(a,c,b) \wedge \neg B(a,d,b)\Big) \vee \Big(B(a,d,b) \wedge \neg B(a,c,b)\Big) \text{ } \textit{Separation}$$

1.3. **The first results: Frasnay, Thomas.** The first complete description of a nontrivial definability lattice was, apparently, the work of C. Frasnay (1965), where he foubd the lattice for the order of rational numbers: the (spaces of) Huntington's relations, to which the order itself and the equality are added.

The order of rational numbers is a homogeneous structure.

Another famous homogeneous structure is the random (universal, generic) graph (unoriented). It was investigated by S. Thomas [Tho91]. Let us for $k = 2,3,4,5$ define $R^{(k)}$ as elements $x_1,...,x_k$ are pairwise distinct with odd number of edges between them.

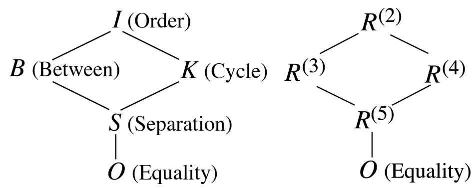

FIGURE 1. The Lattices for: $\langle \mathbb{Q}, \{<\}\rangle$; for the random graph

The definability latticies for the two structures are presented at Fig.1.

Thomas Conjecture [Tho91] The lattice of every homogeneous f.g. definability space is finite.

1.4. **Svenonius Theorem – the Key Instrument.**

Svenonius Theorem. [Sve59] *Let S^-, S be countable definability spaces where $S^- \subset S$. Then, the following statements are equivalent for every relation $R \in S$:*

(a) $R \notin S^-$; and

(b) There are: a countable elementary extension S' of S and a permutation φ on the universe of S' such that φ preserves all relations from S^- and does not preserve the relation R.

So, the theorem of Svenonius states that if we consider not only subgroups of permutations of the original space, but also permutations of its elementary extensions then the mapping from subspaces to (closed) groups of automorphisms is injective.

As R. Buchi anf K. Danhof said the theorem is the basis for Erlangen Program of Defenability theory.

Permutation groups were used for description of definability lattices in many cases (see e.g. [Mac11]), mostly for homogeneous structures: these structures are ω-categorical.

The next step seems natural when the extensions required by the Svenonius theorem are observable.

A countable structure $\mathcal{M}$ is called *upward complete* , if $\mathcal{M} \cong \mathcal{M}'$ for every countable elementary extension of $\mathcal{M}' \succ \mathcal{M}$. An upward complete structure elementary equivalent to a structure is called completion of the latter [SS22].

For upward complete structures, the Svenonius theorem 1.4 has a simple form:

Corollary 1. *Let S^-, S be countable definability spaces on a universe A and $S^- \subset S$, the structure with the universe A and space S is upward complete. Then the following (a) and (b) are equivalent for every relation $R \in S$.*

(a) $R \in S^-$.

(b) The group of permutations on A, preserving S^-, preserves R.

We will call a graph G *discretely homogeneous* if it is connected and for each radius r, all neighbourhoods, with the exception of finitely many, of radius r are isomorphic (as oriented labelled graphs with a pointed neighbourhood centre of it).

As shown in [SS22], every discretely homogeneous structure has a complete upward extension with a fairly simple automorphism group.

A natural example of a discretely homogeneous structure is $\langle \mathbb{Z}, \{'\}\rangle$, where ' is the successor relation ($x' = x + 1$).

For every positive integer n we have the relation D_n 'to be in distance n' and the relation E_n of 'n-codirectional' or 'n-equipollence' of segments:

$$D_n(a,b) \rightleftharpoons |a-b| = n$$

$$E_n(a,b,c,d) \rightleftharpoons a-b = c-d = n \vee a-b = c-d = -n$$

In [SS21] we proved that the (infinite) definability lattice for $\langle \mathbb{Z},' \rangle$ consists of the spaces of $=$, $+n, D_n, E_n$, for all positive integers n.

A nanural next step is $\langle \mathbb{Z}, \{<\} \rangle$. There are spaces B, C, S, all different.

We can prove that the definability lattice inbetween $<$ and D_1 is

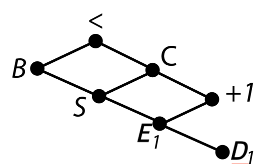

Figure 2. Fragment of $\langle \mathbb{Z}, \{<\} \rangle$ lattice

Open Problem 1. The definability lattice for $\langle \mathbb{Z}, \{<\} \rangle$ containes $<$, B, C, S and the lattice of $\langle \mathbb{Z}, \{'\} \rangle$. Is there anything else?

Let us consider discretely homogeneous structures being natural extensions of $\langle \mathbb{Z}, \{'\} \rangle$.

The first is 'infinite checkered paper' with the vertical and horizontal shifts: $\langle \mathbb{Z} \times \mathbb{Z}, \{\uparrow, \rightarrow\} \rangle$.

Open Problem 2. Describe the definability lattice for $\langle \mathbb{Z} \times \mathbb{Z}, \{\uparrow, \rightarrow\} \rangle$. A simplified case: $\langle \mathbb{Z} \times \mathbb{Z}, \{D_1\} \rangle$.

The second example is an infinite non-rooted non-oriented tree with all vertices of degree 3; $D_n(a,b)$ is true when the length of the shortest path between a and b is n. So, D_1 is the original relation. It is easy to see, that we have the sublattice:

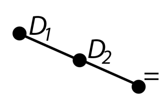

Figure 3. Fragment of the lattice for the infinite tree

Open Problem 3. Is there anything else?

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