

Proof complexity of Mal'tsev CSPs

Azza Gaysin

azza.gaysin@gmail.com

Abstract. Constraint Satisfaction Problems (CSPs) form a broad class of combinatorial problems, which can be formulated as homomorphism problems between relational structures. The CSP dichotomy theorem classifies all such problems over finite domains into two categories: NP-complete and polynomial-time [1], [2].

Polynomial-time CSPs can be further subdivided into smaller subclasses. Mal'tsev CSPs are defined by the property that every relation in the problem is invariant under a Mal'tsev operation, a ternary operation m satisfying the identities $m(x, y, y) = m(y, y, x) = x$ for all x, y . Bulatov proved that Mal'tsev CSPs are solvable in polynomial time [3], and Dalmau proposed an algorithm for such CSPs based on compact sets of representatives [4].

For satisfiable instances, Dalmau's algorithm produces a solution that can be independently verified in polynomial time. However, for unsatisfiable instances, the only evident polynomial-size certificate of the algorithm's correctness is the computation trace of the algorithm on the given instance.

The negation of an unsatisfiable CSP instance can be expressed as a propositional tautology. By applying proof complexity techniques, we extend Dalmau's algorithm with an independent polynomial-size witness of its correctness for negative answers as well. We formalize the algorithm within the framework of bounded arithmetic V^1 , which captures polynomial-time reasoning and corresponds to the extended Frege proof system. We show that V^1 proves the soundness of Dalmau's algorithm, implying that tautologies expressing the non-existence of a solution for unsatisfiable instances admit short extended Frege proofs.

References

1. Zhuk, D.: A proof of the CSP dichotomy conjecture. *J. ACM* **67**(5), 1–78 (2020).
2. Bulatov, A.: A dichotomy theorem for nonuniform CSPs. In: 2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS), pp. 319–330. IEEE, Berkeley, CA, USA (2017)
3. Bulatov, A.: Mal'tsev constraints are tractable. Technical Report PRG-02-05, Computing Laboratory, Oxford University, (2002).
4. Dalmau, V.: Mal'tsev constraints made simple. Electronic Colloquium on Computational Complexity, Report No. 97 (2004)