"PARAMETER FREE Π_1–INDUCTION AND RESTRICTED EXPONENTIATION" AND "A MODEL THEORETIC APPROACH TO PARAMETER FREE Π_2–INDUCTION" (EXTENDED ABSTRACT)

A. Cordón–Franco, A. Fernández–Margarit, and F.F. Lara–Martín

Dpto. Ciencias de la Computación e Inteligencia Artificial
Facultad de Matemáticas. Universidad de Sevilla
C/ Tarfia, s/n, 41012 Sevilla (Spain)
{acordon,afmargarit,fflara}@us.es

May 30, 2009

Abstract. This work falls into two parts. In the first part, we characterize the sets of all Π_2 and all $\mathcal{B}(\Sigma_1)$ (= boolean combinations of Σ_1) theorems of $I\Pi_1^-$ in terms of restricted exponentiation; and use these characterizations to prove that both sets are not deductively equivalent. In the second part, we investigate how these results generalize to $I\Pi_2^-$. We present a new model–theoretic approach to $I\Pi_2^-$ that allows us to obtain informative characterizations of the set of all Π_2 and all $\mathcal{B}(\Sigma_2)$ theorems of $I\Pi_2^-$; and to give a model–theoretic proof of a conservation result by L. Beklemishev's stating that $I\Pi_2^-$ is a $\mathcal{B}(\Sigma_2)$ conservative extension of $I\Sigma_1^-$.

1. Introduction

A natural question in studying the strength of an arithmetic theory T, is to characterize the sets $Th_\Gamma(T)$ of all theorems of T of a specifical logical complexity Γ. In this paper we shall deal with this question for the scheme of parameter free Π_{n+1}–induction $I\Pi_{n+1}^-$.

$I\Pi_{n+1}^-$ is the first order theory in the language of Peano Arithmetic $\mathcal{L} = \{0, 1, +, \cdot, \leq\}$ given by P^- together with the *induction scheme*

$$(I_\varphi) \quad \varphi(0) \wedge \forall x\,(\varphi(x) \rightarrow \varphi(x+1)) \;\rightarrow\; \forall x\,\varphi(x)$$

for each $\varphi(x) \in \Pi_{n+1}^-$. As usual, we write $\varphi(x) \in \Gamma^-$ to mean that φ is in Γ and contains no other free variables but x (we refer the reader to [9], [11] for the fundamental notions and results of first order Arithmetic, and to [3], [8], [10] for properties of $I\Pi_{n+1}^-$).

This paper falls naturally into two parts.

Partially supported by grant MTM2008–06435, Ministerio de Educación y Ciencia, Spain.

Part I: Parameter Free Π_1–Induction and Restricted Exponentiation

In the first part, we concentrate in the case $n = 0$, with the aim of separating the sets of all Π_2 and all $\mathcal{B}(\Sigma_1)$ (boolean combinations of Σ_1) theorems of $I\Pi_1^-$. Our interest in $I\Pi_1^-$ is motivated by the fact that, in a sense, this theory lies in between *weak* and *strong* fragments of Arithmetic. First, observe that $I\Pi_1^-$ is axiomatized by a set of true Σ_2 sentences and hence follows from the set of all true Π_1 arithmetic sentences, $Th_{\Pi_1}(\mathcal{N})$. So, $I\Pi_1^-$ cannot prove the totality of functions of more than polynomial growth and in particular exponentiation is not necessarily total in a model of $I\Pi_1^-$. From this point of view, $I\Pi_1^-$ can be considered a *weak* theory close to the scheme of induction for bounded formulas, $I\Delta_0$. In fact, it is a theorem of T. Bigorajska that $I\Pi_1^-$ satisfies Parikh's theorem *"asymptotically."*

Theorem 1.1 *(Bigorajska,* [4]*).* *Suppose $\varphi(x, y) \in \Sigma_1$ and $I\Pi_1^- \vdash \forall x\, \exists y\, \varphi(x, y)$. Then, there is a term $t(x)$ such that $I\Pi_1^- \vdash \exists u\, \forall x > u \exists y \leq t(x)\, \varphi(x, y)$.*

Since [10] it has been known, however, that there is a tight relationship between $I\Pi_1^-$ and the *strong* system $I\Delta_0 + exp$ (as usual exp denotes a Π_2 axiom expressing "exponentiation is total"). Although both theories are incomparable with respect to inclusion, they share the same $\mathcal{B}(\Sigma_1)$ consequences.

Theorem 1.2 *(Kaye–Paris–Dimitracopoulos,* [10]*).*

(1) *$I\Delta_0 + exp$ is conservative over $I\Pi_1^-$ with respect to Σ_2 sentences.*
(2) *$I\Pi_1^-$ is conservative over $I\Delta_0 + exp$ with respect to Π_2 sentences.*
(3) *So, $Th_{\Pi_1}(I\Pi_1^-) = Th_{\Pi_1}(I\Delta_0 + exp)$ and $Th_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-) = Th_{\mathcal{B}(\Sigma_1)}(I\Delta_0 + exp)$.*

As a result $I\Pi_1^-$ proves that there are unboundedly many primes and implies the Δ_0 pigeonhole principle (for they are Π_1 consequences of $I\Delta_0 + exp$). Whether or not these two properties are also provable in $I\Delta_0$ are important open problems. Besides, the following hierarchy of theories between $I\Delta_0$ and $I\Delta_0 + exp$ raises:

$$I\Delta_0 \subseteq Th_{\Pi_1}(I\Pi_1^-) \subseteq Th_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-) \subseteq Th_{\Pi_2}(I\Pi_1^-) \subset I\Delta_0 + exp$$

In [14], [15] A. Wilkie and J. Paris studied the Π_1 theorems of $I\Pi_1^-$ under its equivalent formulation of $Th_{\Pi_1}(I\Delta_0 + exp)$. In [14] Wilkie characterized the Π_1 sentences provable in $I\Delta_0 + exp$ as those Π_1 sentences which are interpretable in Robinson's Arithmetic Q. In [15] the authors showed that a Π_1 sentence expressing the Herbrand consistency of Q is provable in $I\Delta_0 + exp$ but not in $I\Delta_0$ and hence $Th_{\Pi_1}(I\Pi_1^-)$ is strictly stronger than $I\Delta_0$. In contrast, no similar results were known for the $\mathcal{B}(\Sigma_1)$ and the Π_2 theorems of $I\Pi_1^-$. In particular, it was not known whether $Th_{\Pi_2}(I\Pi_1^-)$ follows from $Th_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$.

The aim of the first part of the paper is to fill these gaps in our understanding of the hierarchy above

- by obtaining informative axiomatizations of $Th_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$ and $Th_{\Pi_2}(I\Pi_1^-)$, and
- by proving that the inclusions are all strict.

To this end, the key idea is to consider certain forms of *restricted exponentiation*. Although $I\Pi_1^-$ does not prove exp, we shall show that $I\Pi_1^-$ *can* prove that x^y exists whenever x, y are restricted to be in certain initial segments determined by the Σ_1 *definable elements* of a model. Namely, given $\mathfrak{A} \models I\Delta_0$ we shall deal with the following submodels of $\mathfrak{A}$:

- $\mathcal{K}_1(\mathfrak{A})$ is the submodel of $\mathfrak{A}$ whose domain consists of all Σ_1 definable elements; $\mathcal{I}_1(\mathfrak{A})$ is the least initial segment of $\mathfrak{A}$ containing $\mathcal{K}_1(\mathfrak{A})$.
- $\mathcal{K}_1^1(\mathfrak{A}) = \mathcal{K}_1(\mathfrak{A}, \mathcal{I}_1(\mathfrak{A}))$, i.e. the submodel of $\mathfrak{A}$ whose domain consists of all elements which are Σ_1 definable with a parameter from $\mathcal{I}_1^1(\mathfrak{A})$; $\mathcal{I}_1^1(\mathfrak{A})$ is the least initial segment of $\mathfrak{A}$ containing $\mathcal{K}_1^1(\mathfrak{A})$.

Considering these submodels will allow us to isolate the exact amount of exponentiation available in $I\Pi_1^-$: x^y exists whenever the base x is in $\mathcal{I}_1^1(\mathfrak{A})$ and the exponent y is in $\mathcal{I}_1(\mathfrak{A})$. In addition, the $\mathcal{B}(\Sigma_1)$ and the Π_2 theorems of $I\Pi_1^-$ can be characterized over $I\Delta_0$ by principles of the form "*For certain x, y, x^y exists.*" More precisely, next theorem collects together the main results of the first part of the paper.

Theorem 1.3.

(1) $I\Pi_1^- \vdash \forall a \in \mathcal{I}_1(\mathfrak{A})\ \forall b \in \mathcal{I}_1^1(\mathfrak{A})\ \exists y\,(y = b^a)$.
(2) $Th_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-) \equiv I\Delta_0 + \forall a, b \in \mathcal{I}_1(\mathfrak{A})\ \exists y\,(y = b^a)$.
(3) $Th_{\Pi_2}(I\Pi_1^-) \equiv I\Delta_0 + \forall a \in \mathcal{I}_1(\mathfrak{A})\ \forall b \in \mathcal{I}_1^1(\mathfrak{A})\ \exists y\,(y = b^a)$.
(4) $Th_{\Pi_1}(I\Pi_1^-) \nvdash Th_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$, *and* $Th_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-) \nvdash Th_{\Pi_2}(I\Pi_1^-)$.

Let us observe that the characterizations above are to be seen as axiomatizations in the language $\mathcal{L} = \{0, 1, +, \cdot, \leq\}$. Indeed, a quantifier of the form "$\forall x \in \mathcal{I}_1(\mathfrak{A})$" in front of a formula $\Psi(x)$ is intended as a shorthand for the scheme:

$$\{\forall x, y\,(\delta(y) \wedge \forall y'(\delta(y') \rightarrow y' = y) \wedge x \leq y \ \rightarrow\ \Psi(x)) \,:\, \delta \in \Sigma_1\}$$

In a similar vein, a quantifier of the form "$\forall x \in \mathcal{I}_1^1(\mathfrak{A})$" in front of a formula $\Psi(x)$ unravels to:

$$\{\forall x, y, z, u \left(\begin{array}{c} \delta_1(y) \wedge \forall y'(\delta_1(y') \rightarrow y' = y) \wedge z \leq y \\ \delta_2(u, z) \wedge \forall u'(\delta_2(u', z) \rightarrow u' = u) \wedge x \leq u \end{array} \ \rightarrow\ \Psi(x) \right) \,:\, \delta_1, \delta_2 \in \Sigma_1\}$$

Our proofs are model–theoretic and are based on the proofs of Theorem 1.2 and other related results in [8] and [10]. It will be important, however, to make explicit a property that appeared only implicitly in [8] or [10]: although $I\Pi_1^-$ is much weaker than the *full* scheme of induction for Σ_1 formulas $I\Sigma_1$, $I\Pi_1^-$ allows for some *restricted* Σ_1 induction: induction up to $\mathcal{I}_1(\mathfrak{A})$ for Σ_1 formulas with parameters in $\mathcal{I}_1(\mathfrak{A})$. This property clarifies why a certain amount of exponentiation is available in $I\Pi_1^-$ and, in a sense, explains the particular behavior of $I\Pi_1^-$ mixing features of weak and strong fragments of Peano Arithmetic.

Part II: A Model Theoretic Approach to Parameter Free Π_2–Induction

In the second part, we investigate how previous results generalize to $I\Pi_{n+1}^-$ for $n > 0$. It turns out that the ideas in the first part apply equally well to theories $I\Sigma_n + I\Pi_{n+1}^-$ for $n > 0$. However, it seems not to be the case for theories $I\Pi_{n+1}^-$. So, new techniques are needed to study the Π_{n+2} and the $\mathcal{B}(\Sigma_{n+1})$ consequences of $I\Pi_{n+1}^-$ for $n > 0$.

First of all, it is worth noticing that in [3] the set of all $\mathcal{B}(\Sigma_{n+1})$ theorems of $I\Pi_{n+1}^-$ was succinctly characterized as follows:

Theorem 1.4 *(Beklemishev,* [3]*).* *For each $n > 0$:*

(1) *$I\Pi^-_{n+1}$ is conservative over $I\Sigma^-_n$ with respect to $\mathcal{B}(\Sigma_{n+1})$ sentences.*
(2) *So, $Th_{\mathcal{B}(\Sigma_{n+1})}(I\Pi^-_{n+1}) \equiv I\Sigma^-_n$.*

However, it is an open question whether Beklemishev's conservation result can be extended to Π_{n+2} sentences. Put slightly different, informative axiomatizations of the set $Th_{\Pi_{n+2}}(I\Pi^-_{n+1})$ are not known for $n > 0$. In addition, Beklemishev's proof of Theorem 1.4 is by means of proof–theoretic tools. Somewhat surprisingly, no model–theoretic proofs of this result are known. In particular, even after Kaye's excellent work on parameter free schemes in his Ph.D. thesis [8], we do not know how to prove *model–theoretically* that the provably total computable functions of $I\Pi^-_2$ are, precisely, the primitive recursive functions (observe that this fact follows from Theorem 1.4 for $n = 1$).

Motivated by these facts, we shall present new model–theoretic techniques to investigate theories $I\Pi^-_{n+1}$ that will allow us to obtain:

- An alternative model–theoretic approach to $I\Pi^-_1$. We obtain a new characterization of $Th_{\mathcal{B}(\Sigma_1)}(I\Pi^-_1)$ and make use of it to give a new proof of Theorem 1.2.
- Informative axiomatizations of $Th_{\Pi_2}(I\Pi^-_2)$ and of $Th_{\mathcal{B}(\Sigma_2)}(I\Pi^-_2)$.
- A model–theoretic proof of Theorem 1.4 for $n = 1$ (and so we give a *model–theoretic* solution to Kaye's question asking whether the provably total computable functions of $I\Pi^-_2$ are exactly the primitive recursive ones).

Let us observe that we shall concentrate in the schemes $I\Pi^-_1$ and $I\Pi^-_2$, although it seems to be more or less routine to extend the methods and results presented in this second part to $I\Pi^-_{n+1}$ for every $n \geq 0$.

The key idea is to carry out an analysis of $I\Pi^-_1$ through the related scheme of Σ_1–induction up to Σ_1–definable elements. If Γ is a set of formulas and $n \geq 0$, we denote by $I(\Gamma, \mathcal{K}_{n+1})$ the theory given by $I\Sigma^-_n$ together with the induction scheme

$$\{\varphi(0) \wedge \forall x\,(\varphi(x) \to \varphi(x+1)) \to \forall x \in \mathcal{K}_{n+1}(\mathfrak{A})\,\varphi(x) :\ \varphi(x) \in \Gamma\}$$

As noted before, a quantifier of the form "$\forall x \in \mathcal{K}_{n+1}(\mathfrak{A})$" must be intended as the corresponding scheme in the the language of Arithmetic. The parameter free version of this theory will be denoted by $I(\Gamma^-, \mathcal{K}_{n+1})$. Interestingly, $I(\Sigma^-_1, \mathcal{K}_1)$ turns out to be equivalent to $I\Pi^-_1$; and this suggests the existence of a tied relation between $I\Pi^-_1$ and the stronger theory $I(\Sigma_1, \mathcal{K}_1)$. This last theory is of interest to us because it is more amenable to a direct characterization of the set of all its Π_2–consequences in terms of a closure operation under an *inference rule* associated with it (a similar characterization was obtained by Ch. Parsons for $I\Sigma_1$ and Σ_1–IR in [12]).

Indeed, we also introduce the inference rule $(\Gamma, \mathcal{K}_{n+1})$–IR given by

$$\frac{\varphi(0) \wedge \forall x\,(\varphi(x) \to \varphi(x+1))}{\forall x \in \mathcal{K}_{n+1}(\mathfrak{A})\,\varphi(x)}$$

(where $\varphi(x) \in \Gamma$) and study the relationship among the rules obtained for $\Gamma = \Sigma_1$, Π_2 and $n = 0$, 1, on the one hand; and the fragment $I\Pi^-_1$ and the parameter free versions of the rules, on the other. The main results obtained for $(\Sigma_1, \mathcal{K}_1)$–IR are the following (here, for a theory T and an inference rule R, $T + R$ denotes the closure of T under first order logic

and nested applications of R; and $[T, R]$ denotes the closure of T under first order logic and unnested applications of R):

Proposition 1.5.

(1) *$I\Pi_1^-$ is Π_2–conservative over $I\Delta_0 + (\Sigma_1, \mathcal{K}_1)$–IR.*
(2) *The following theories are equivalent:*
   (a) $I\Delta_0 + (\Pi_2, \mathcal{K}_1)$–IR
   (b) $I\Delta_0 + (\Sigma_1, \mathcal{K}_1)$–IR
   (c) $[I\Delta_0, (\Sigma_1, \mathcal{K}_1)$–IR$]$

The parameter free version of $(\Sigma_1, \mathcal{K}_1)$–IR turns out to be equivalent to the parameter free version of the rule Π_1–IR$_0$:

$$\frac{\varphi(0) \wedge \forall x\, (\varphi(x) \to \varphi(x+1))}{\varphi(0) \to \forall x\, \varphi(x)}$$

where $\varphi(x) \in \Pi_1$. It is well–known that this rule is equivalent to Σ_1–IR (in fact, both rules are *congruent*, see [2]). However, while Σ_1–IR is equivalent (congruent) to its parameter free counterpart, Π_1–IR$_0$ is strictly stronger than Π_1^-–IR$_0$. Our approach to Π_1^-–IR$_0$ provides an in depth analysis of this rule that clarifies its relation to $I\Pi_1^-$ and to the weaker rule Π_1–IR. Next proposition sums up the results obtained:

Proposition 1.6.

(1) *$[I\Delta_0, \Pi_1$–$IR_0]$ is Σ_2–conservative over $[I\Delta_0, \Pi_1^-$–$IR_0]$.*
(2) *The following theories are equivalent:*
   (a) *$I\Delta_0 + \Pi_1^-$–IR_0.*
   (b) *$[I\Delta_0, \Pi_1^-$–$IR_0]$.*
   (c) *$[I\Delta_0, (\Sigma_1^-, \mathcal{K}_1)$–$IR]$*
(3) *$I\Delta_0 + \Pi_1^-$–IR_0 is Π_1–conservative over $I\Delta_0 + \Pi_1$–IR.*

As a corollary, Proposition 1.5 provides a, very natural and clear, new proof of part (2) in Theorem 1.2 (Π_2 conservation between $I\Pi_1^-$ and $I\Delta_0$+exp) and, together with Proposition 1.6, it enables us to derive new characterizations of $Th_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$ and $Th_{\Pi_1}(I\Pi_1^-)$:

Theorem 1.7.

(1) *$Th_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-) \equiv I\Delta_0 + \Pi_1^-$–$IR_0$.*
(2) *$Th_{\Pi_1}(I\Pi_1^-) \equiv I\Delta_0 + \Pi_1$–IR.*

Propositions 1.5 and 1.6 have counterparts for $I\Pi_2^-$. However, there are important differences between their generalizations. Proposition 1.5 holds for $I\Pi_2^-$ and the rules $(\Sigma_2, \mathcal{K}_2)$–IR and $(\Pi_3, \mathcal{K}_2)$–IR taking $I\Sigma_1$ as our base theory instead of $I\Delta_0$ (in fact, it can be proved that $I\Sigma_1 + I\Pi_2^-$ is Π_2–conservative over $I\Sigma_1 + (\Sigma_2, \mathcal{K}_2)$–IR.) In turn, Proposition 1.6 holds for Π_2^-–IR$_0$, Π_2–IR and $(\Sigma_2^-, \mathcal{K}_2)$–IR over the base theory $I\Sigma_1^-$. This is the key ingredient in our proof of Beklemishev's conservation result in Theorem 1.4, together with some additional properties of Π_2^-–IR$_0$:

Theorem 1.8. *Let T be a $\mathcal{B}(\Sigma_2)$-axiomatizable extension of $I\Delta_0$. Then:*

(1) *$T + I\Sigma_1^- \equiv [T, \Pi_2^-$–$IR_0] \equiv T + \Pi_2^-$–$IR_0$.*
(2) *$T + (\Sigma_2, \mathcal{K}_2)$–IR is Π_2–conservative over $T + \Pi_2^-$–IR_0.*
(3) *$T + I\Pi_2^-$ is $\mathcal{B}(\Sigma_2)$–conservative over $T + \Pi_2^-$–IR_0.*

Our proofs are model–theoretic. For conservation results we use the model–theoretic framework introduced by J. Avigad in [1] as well as properties of Σ_2–closed models (a hierarchical generalization in the setting of first order Arithmetic of the classic notion of an *existentially closed model*). In the analysis of the rules $(\Sigma_m, \mathcal{K}_{n+1})$–IR, we exploit the relation between restricted induction and a certain form of *restricted iteration*. Restricted iteration plays here the role of restricted exponentiation in the first part of the paper.

References

[1] Avigad, J. *Saturated models of universal theories*. Annals of Pure and Applied Logic, 118:219–234, 2002.

[2] Beklemishev, L.D. *Induction rules, reflection principles and provably recursive functions*. Annals of Pure and Applied Logic, 85(3):193–242, 1997.

[3] Beklemishev, L. *Parameter free induction and provably total computable functions*. Theoretical Computer Science, 224:13–33, 1999.

[4] Bigorajska, T. *On Σ_1–definable functions provably total in $I\Pi_1^-$*. Mathematical Logic Quartely, 41:135–137, 1995.

[5] Cordón–Franco, A; Fernández–Margarit, A.; Lara–Martín, F.F. *On the quantifier complexity of $\Delta_{n+1}(T)$–induction*. Archive for Mathematical Logic, 43:371–398, 2004.

[6] Cordón–Franco, A; Fernández–Margarit, A.; Lara–Martín, F.F. *Fragments of Arithmetic and true sentences*. Mathematical Logic Quartely, 51:313–328, 2005.

[7] Fernández–Margarit, A.; Lara–Martín, F.F. *Induction, Minimization and Collection for $\Delta_{n+1}(T)$–formulas*. Archive for Mathematical Logic, 43:505–542, 2004.

[8] Kaye, R. *Diophantine and Parameter–free Induction*. Ph.D. Thesis. University of Manchester, 1987.

[9] Kaye, R. *Models of Peano Arithmetic*. Oxford Logic Guides, 15. Clarendon Press. Oxford, 1991.

[10] Kaye, R.; Paris, J; Dimitracopoulos, C. *On parameter free induction schemas*. The Journal of Symbolic Logic, 53(4):1082–1097, 1988.

[11] Hájek, P.; Pudlák, P. *Metamathematics of First–Order Arithmetic*. Perspectives in Mathematical Logic, Springer Verlag, 1993.

[12] Parsons, C.: *On n–quantifier induction*. The Journal of Symbolic Logic, 37(3):466–482, 1972.

[13] Wilkie, A. *Some Results and Problems on Weak Systems of Arithmetic*. Logic Colloquium '77, pp. 285–296. North Holland, 1978.

[14] Wilkie, A. *On sentences interpretable in systems of arithmetic*. Logic Colloquium '84, pp. 329–342, North–Holland, 1986.

[15] Wilkie, A; Paris, J. *On the scheme of induction for bounded arithmetic formulas*. Annals of Pure and Applied Logic, 25:261–302, 1987.