On Recognizable Languages of Infinite Pictures

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Pictures

Pictures are two-dimensional words.

Let Σ be a finite alphabet and $\# \notin \Sigma$ and let $\hat{\Sigma} = \Sigma \cup \{\#\}$. If m and n are two integers > 0 or if $m = n = 0$, a picture of size (m, n) over Σ is a function p from $\{0, 1, \ldots, m+1\} \times \{0, 1, \ldots, n+1\}$ into $\hat{\Sigma}$ such that:

$p(0, i) = p(m+1, i) = \#$ for all integers $i \in \{0, 1, \ldots, n+1\}$ and $p(i, 0) = p(i, n+1) = \#$ for all integers $i \in \{0, 1, \ldots, m+1\}$ and $p(i, j) \in \Sigma$ if $i \notin \{0, m+1\}$ and $j \notin \{0, n+1\}$.

$$
\begin{array}{l}
\#\#\#\#\# \\
\#\,a\;a\;b\,\# \\
\#\,b\;c\;a\,\# \\
\#\#\#\#\#
\end{array}
$$

picture of size $(3, 2)$

Acceptance of Pictures

For a language L of finite pictures, the following statements are equivalent:

Theorem

- *L is accepted by a four-way automaton.*
- *L is accepted by a finite tiling system.*
- *L is definable in existential monadic second order logic EMSO in the signature $((P_a)_{a\in\Sigma}, S_1, S_2)$.*

Acceptance of Infinite Words

Acceptance of infinite words by finite automata was firstly considered by **Büchi** in the sixties in order to study the decidability of the monadic second order theory S1S of one successor over the integers.

Acceptance of Infinite Pictures

Acceptance of infinite pictures by finite tiling systems is a generalization of:

1. Acceptance of infinite words by automata.
2. Acceptance of finite pictures by tiling systems.

Infinite Pictures

An ω-picture over Σ is a function p from $\omega \times \omega$ into $\hat{\Sigma}$ such that $p(i,0) = p(0,i) = \#$ for all $i \geq 0$ and $p(i,j) \in \Sigma$ for $i,j > 0$.

$$
\begin{array}{l}
\vdots \\
\# \\
\#b \quad \cdots \\
\#a\ a \\
\#b\ c\ a \\
\#\#\#\#\#\ldots
\end{array}
$$

The set $\Sigma^{\omega,\omega}$ of ω-pictures over Σ is a strict subset of the set $\hat{\Sigma}^{\omega^2}$ of functions from $\omega \times \omega$ into $\hat{\Sigma}$.

Tiling Systems

A tiling system is a tuple $\mathcal{A}=(Q,\Sigma,\Delta)$, where Q is a finite set of states, Σ is a finite alphabet, $\Delta \subseteq (\hat{\Sigma} \times Q)^4$ is a finite set of tiles.

A Büchi tiling system is a pair $(\mathcal{A},F)$ where $\mathcal{A}=(Q,\Sigma,\Delta)$ is a tiling system and $F \subseteq Q$ is the set of accepting states.

Tiles

Tiles are denoted by
$\begin{pmatrix} (a_3, q_3) & (a_4, q_4) \\ (a_1, q_1) & (a_2, q_2) \end{pmatrix}$ with $a_i \in \hat{\Sigma}$ and $q_i \in Q$,

and in general, over an alphabet Γ, by
$\begin{pmatrix} b_3 & b_4 \\ b_1 & b_2 \end{pmatrix}$ with $b_i \in \Gamma$.

A combination of tiles is defined by:

$$\begin{pmatrix} b_3 & b_4 \\ b_1 & b_2 \end{pmatrix} \circ \begin{pmatrix} b'_3 & b'_4 \\ b'_1 & b'_2 \end{pmatrix} = \begin{pmatrix} (b_3, b'_3) & (b_4, b'_4) \\ (b_1, b'_1) & (b_2, b'_2) \end{pmatrix}$$

Runs of a Tiling System

A run of a tiling system $\mathcal{A}=(Q, \Sigma, \Delta)$ over an ω-picture $p \in \Sigma^{\omega,\omega}$ is a mapping ρ from $\omega \times \omega$ into Q such that for all $(i,j) \in \omega \times \omega$ with $p(i,j) = a_{i,j}$ and $\rho(i,j) = q_{i,j}$ we have

$$\begin{pmatrix} a_{i,j+1} & a_{i+1,j+1} \\ a_{i,j} & a_{i+1,j} \end{pmatrix} \circ \begin{pmatrix} q_{i,j+1} & q_{i+1,j+1} \\ q_{i,j} & q_{i+1,j} \end{pmatrix} \in \Delta.$$

Acceptance by Tiling Systems

Definition (Altenbernd, Thomas, Wöhrle 2002)

Let $\mathcal{A}=(Q,\Sigma,\Delta)$ be a tiling system, $F \subseteq Q$.

- The ω-picture language Büchi-recognized by $(\mathcal{A},F)$ is the set of ω-pictures $p \in \Sigma^{\omega,\omega}$ such that there is some run ρ of $\mathcal{A}$ on p and $\rho(v) \in F$ for for infinitely many $v \in \omega^2$.

Examples

Let $\Sigma = \{a, b\}$.

- $L_0 = \{b\}^{\omega,\omega}$ is the set of ω-pictures carrying solely label b.
- L_1 is the set of ω-pictures containing at least one letter a.
- L_2 is the set of ω-pictures containing infinitely many letters a.

L_0, L_1, and L_2 are Büchi-recognizable.
[Altenbernd, Thomas, Wöhrle 2002]

Simulation of a Turing Machine

Let $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0)$ be a non deterministic Turing machine and $F \subseteq Q$. The ω-language Büchi accepted by $(\mathcal{M}, F)$ is the set of ω-words $\sigma \in \Sigma^\omega$ such that there exists a run $r = (q_i, \alpha_i, j_i)_{i \geq 1}$ of $\mathcal{M}$ on σ and infinitely many integers i such that $q_i \in F$.
For an ω-language $L \subseteq \Sigma^\omega$ we denote L^B the language of infinite pictures $p \in \Sigma^{\omega,\omega}$ such that the first row of p is in L and the other rows are labelled with the letter B which is assumed to belong to Σ.

Lemma

If $L \subseteq \Sigma^\omega$ is accepted by some Turing machine with a Büchi acceptance condition, then L^B is Büchi recognizable by a finite tiling system.

Simulation of a Turing Machine

We can define a set of tiles Δ in such a way that for $\sigma \in \Sigma^\omega$, a run ρ of the tiling system $\mathcal{T}=(\Sigma, \Gamma \cup Q, \Delta, F)$ over the infinite picture σ^B satisfies:

for each integer $i \geq 0$ $\quad \rho(0,i).\rho(1,i).\rho(2,i)\ldots = \alpha_i = u_i.q_i.v_i$

i.e. $\rho(0,i).\rho(1,i).\rho(2,i)\ldots$ is the $(i+1)^{th}$ configuration of T reading the ω-word $\sigma \in \Sigma^\omega$.

Thus the Büchi tiling system $(\mathcal{T},F)$ recognizes the language L^B.

Closure Properties

Theorem (Altenbernd, Thomas, Wöhrle 2002)

The class of ω-picture languages which are Büchi-recognized by a tiling system is closed under finite union and finite intersection.

Closure under union follows from the non-deterministic behaviour of tiling systems.

Closure under intersection follows from classical product constructions.

Non Closure under Complementation

Theorem (Altenbernd, Thomas, Wöhrle 2002)

The class of ω-picture languages which are Büchi-recognized by a tiling system is not closed under complementation.

The proof can be deduced from the topological complexity of Büchi recognizable languages of infinite pictures.

Topology on Σ^ω

The natural prefix metric on the set Σ^ω of ω-words over Σ is defined as follows:

For $u, v \in \Sigma^\omega$ and $u \neq v$ let

$$\delta(u, v) = 2^{-n}$$

where n is the least integer such that:

the $(n+1)^{st}$ letter of u is different from the $(n+1)^{st}$ letter of v.

This metric induces on Σ^ω the usual Cantor topology for which :

- *open subsets* of Σ^ω are in the form $W.\Sigma^\omega$, where $W \subseteq \Sigma^\star$.
- *closed subsets* of Σ^ω are complements of *open subsets* of Σ^ω.

Borel Hierarchy

Σ^0_1 is the class of open subsets of Σ^ω,

Π^0_1 is the class of closed subsets of Σ^ω,

for any integer $n \geq 1$:

Σ^0_{n+1} is the class of countable unions of Π^0_n-subsets of Σ^ω.

Π^0_{n+1} is the class of countable intersections of Σ^0_n-subsets of Σ^ω.

Π^0_{n+1} is also the class of complements of Σ^0_{n+1}-subsets of Σ^ω.

Borel Hierarchy

The Borel hierarchy is also defined for levels indexed by countable ordinals.

For any countable ordinal $\alpha \geq 2$:

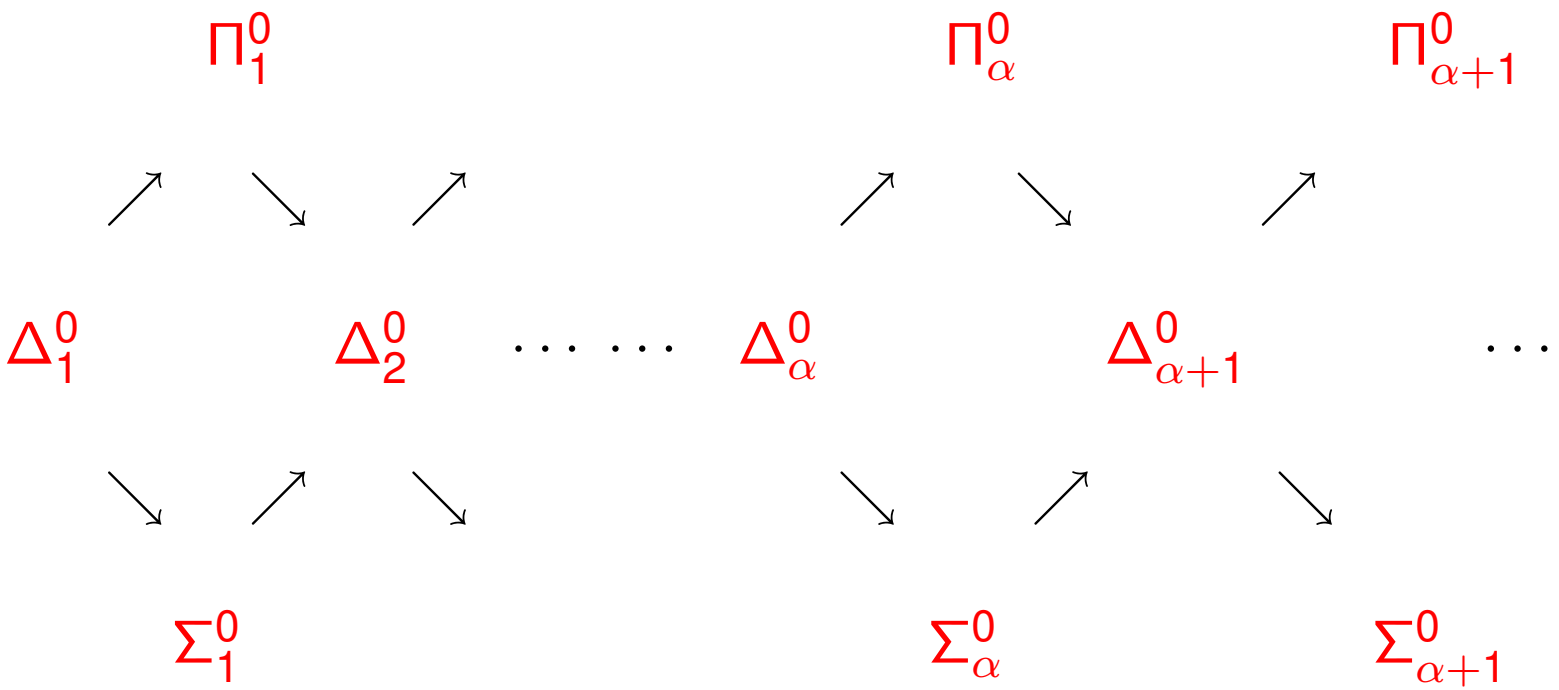
Σ^0_α is the class of countable unions of subsets of Σ^ω in $\bigcup_{\gamma<\alpha} \Pi^0_\gamma$.

Π^0_α is the class of complements of Σ^0_α-sets

$\Delta^0_\alpha = \Pi^0_\alpha \cap \Sigma^0_\alpha$.

Borel Hierarchy

Below an arrow $\longrightarrow$ represents a strict inclusion between Borel classes.



A set $X \subseteq \Sigma^\omega$ is a Borel set iff it is in $\bigcup_{\alpha<\omega_1} \Sigma^0_\alpha = \bigcup_{\alpha<\omega_1} \Pi^0_\alpha$ where ω_1 is the first uncountable ordinal.

Beyond the Borel Hierarchy

There are some subsets of Σ^ω which are not Borel. Beyond the Borel hierarchy is the projective hierarchy.

The class of Borel subsets of Σ^ω is strictly included in the class Σ^1_1 of analytic sets which are obtained by projection of Borel sets.

A set $E \subseteq \Sigma^\omega$ is in the class Σ^1_1 iff :

$$\exists F \subseteq (\Sigma \times \{0,1\})^\omega \text{ such that } F \text{ is } \Pi^0_2 \text{ and}$$

$$E \text{ is the projection of } F \text{ onto } \Sigma^\omega$$

A set $E \subseteq \Sigma^\omega$ is in the class Π^1_1 iff $\Sigma^\omega - E$ is in Σ^1_1.

Suslin's Theorem states that : Borel sets $= \Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$

Complete Sets

A set $E \subseteq \Sigma^\omega$ **is $\mathcal{C}$-complete**, where $\mathcal{C}$ is a Borel class Σ^0_α or Π^0_α

or the class Σ^1_1, **for reduction by continuous functions** iff :

$\forall F \subseteq \Gamma^\omega \quad F \in \mathcal{C}$ iff :

$\exists f$ continuous, $f : \Gamma^\omega \to \Sigma^\omega$ such that $F = f^{-1}(E)$

$(x \in F \leftrightarrow f(x) \in E)$.

Example : $\{\sigma \in \{0,1\}^\omega \mid \exists^\infty i \ \ \sigma(i) = 1\}$ is a Π^0_2-complete-set and it is accepted by a deterministic Büchi automaton.

More Examples of Complete Sets

Examples :

$\{\sigma \in \{0,1\}^\omega \mid \exists i \ \ \sigma(i) = 1\}$ is a Σ_1^0**-complete-set**.

$\{\sigma \in \{0,1\}^\omega \mid \forall i \ \ \sigma(i) = 1\} = \{1^\omega\}$ is a Π_1^0**-complete-set**.

$\{\sigma \in \{0,1\}^\omega \mid \exists^{<\infty} i \ \ \sigma(i) = 1\}$ is a Σ_2^0**-complete-set**.

All these ω-languages are ω-regular.

Topology on $\Gamma^{\omega\times\omega}$

For Γ a finite alphabet having at least two letters, the set $\Gamma^{\omega\times\omega}$ of functions from $\omega \times \omega$ into Γ is usually equipped with the topology induced by the following distance d.

Let x and y in $\Gamma^{\omega\times\omega}$ such that $x \neq y$, then

$$d(x,y) = \frac{1}{2^n} \qquad \text{where}$$

$$n = min\{p \geq 0 \mid \exists (i,j) \;\; x(i,j) \neq y(i,j) \text{ and } i+j = p\}.$$

Then the topological space $\Gamma^{\omega\times\omega}$ is homeomorphic to the topological space Γ^{ω}, equipped with the Cantor topology.

Topology on $\Sigma^{\omega,\omega}$

The set $\Sigma^{\omega,\omega}$ of ω-pictures over Σ, viewed as a topological subspace of $\hat{\Sigma}^{\omega\times\omega}$, is easily seen to be homeomorphic to the topological space $\Sigma^{\omega\times\omega}$,

via the mapping $\varphi : \Sigma^{\omega,\omega} \to \Sigma^{\omega\times\omega}$ defined by $\varphi(p)(i,j) = p(i+1, j+1)$ for all $p \in \Sigma^{\omega,\omega}$ and $i, j \in \omega$.

The Borel hierarchy and analytic sets in $\Sigma^{\omega,\omega}$ are defined as in the Cantor space Γ^{ω}.

Complexity of Some Languages of ω-Pictures

Examples

- $L_0 = \{b\}^{\omega,\omega}$ is the set of ω-pictures carrying solely label b. $L_0 \subseteq \{a,b\}^{\omega,\omega}$ is Π_1^0-complete, it is in $\Pi_1^0 - \Sigma_1^0$.
- $L_1 \subseteq \{a,b\}^{\omega,\omega}$ is the set of ω-pictures containing at least one letter a. L_1 is Σ_1^0-complete, it is in $\Sigma_1^0 - \Pi_1^0$.
- $L_2 \subseteq \{a,b\}^{\omega,\omega}$ is the set of ω-pictures containing infinitely many letters a. L_2 is Π_2^0-complete, it is in $\Pi_2^0 - \Sigma_2^0$.
- $L_3 \subseteq \{a,b\}^{\omega,\omega}$ is the set of ω-pictures containing only finitely many letters a. L_3 is Σ_2^0-complete, it is in $\Sigma_2^0 - \Pi_2^0$.

All these languages are Büchi-recognizable.

Complexity of Büchi-Recognizable Languages of ω-Pictures

Büchi-recognizable languages of infinite pictures have the same complexity as ω-languages of non deterministic Turing machines, or effective analytic sets because :

- If L is accepted by a Büchi Turing machine then L^B is Büchi-recognizable.
- If $L \subseteq \Sigma^{\omega,\omega}$ is Büchi-recognizable then L is definable in existential monadic second order logic. Thus L is an effective analytic set.

Complexity of ω-Languages of Non Deterministic Turing Machines

Non deterministic Büchi or Muller Turing machines accept **effective analytic sets**. The class **Effective-Σ^1_1** of **effective analytic sets** is obtained as the class of **projections of arithmetical sets** and **Effective-Σ^1_1** $\subsetneq \Sigma^1_1$.

Let ω_1^{CK} be the first non recursive ordinal.

Topological Complexity of Effective Analytic Sets

- There are some Σ^1_1-complete sets in **Effective-Σ^1_1**.
- For every non null ordinal $\alpha < \omega_1^{\mathrm{CK}}$, there exists some Σ^0_α-complete and some Π^0_α-complete ω-languages in the class **Effective-Σ^1_1**.
- **(Kechris, Marker and Sami 1989)**
  The supremum of the set of Borel ranks of **Effective-Σ^1_1-sets** is a countable ordinal $\gamma^1_2 > \omega_1^{\mathrm{CK}}$.

Non Closure under Complementation

Theorem (Altenbernd, Thomas, Wöhrle 2002)

The class of ω-picture languages which are Büchi-recognized by a tiling system is not closed under complementation.

Proof. There is a Σ^1_1-complete ω-language $L \subseteq \{0,1\}^\omega$ accepted by a Büchi Turing machine. The ω-picture language L^B is accepted by a Büchi tiling system and is also a Σ^1_1-complete subset of $\{0,1\}^{\omega,\omega}$. Its complement is Π^1_1-complete, so it is not in the class Σ^1_1. Thus it cannot be accepted by any Büchi tiling system.

Decision Problems

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two Büchi tiling systems over the alphabet Σ. Can we decide whether

- $L(\mathcal{T}_1)$ is empty ?
- $L(\mathcal{T}_1)$ is infinite ?
- $L(\mathcal{T}_1) = \Sigma^{\omega,\omega}$?
- $L(\mathcal{T}_1) = L(\mathcal{T}_2)$?
- $L(\mathcal{T}_1) \subseteq L(\mathcal{T}_2)$?
- $L(\mathcal{T}_1)$ is unambiguous ?
- $L(\mathcal{T}_1)$ is Borel ?
- ...

All these problems are highly undecidable, i.e. located beyond the arithmetical hierarchy, in fact at the first or second level of the analytical hierarchy for most of them.

The Analytical Hierarchy

The Analytical Hierarchy is defined for subsets of $\mathbb{N}^l$ where $l \geq 1$ is an integer. It extends the arithmetical hierarchy to more complicated sets.

Theorem

For each integer $n \geq 1$,

(a) $\Sigma^1_n \cup \Pi^1_n \subsetneq \Sigma^1_{n+1} \cap \Pi^1_{n+1}$.

(b) *A set $R \subseteq \mathbb{N}^l$ is in the class Σ^1_n iff its complement is in the class Π^1_n.*

(c) $\Sigma^1_n - \Pi^1_n \neq \emptyset$ *and* $\Pi^1_n - \Sigma^1_n \neq \emptyset$.

The Analytical Hierarchy

Let $k, l > 0$ be some integers and $R \subseteq \mathcal{F}^k \times \mathbb{N}^l$, where $\mathcal{F}$ is the set of all mappings from $\mathbb{N}$ into $\mathbb{N}$.
The relation R is said to be recursive if its characteristic function is recursive.

A subset R of $\mathbb{N}^l$ is analytical if it is recursive or if there exists a recursive set $S \subseteq \mathcal{F}^m \times \mathbb{N}^n$, with $m \geq 0$ and $n \geq l$, such that $(x_1, \ldots, x_l)$ is in R iff

$$(Q_1 s_1)(Q_2 s_2) \ldots (Q_{m+n-l} s_{m+n-l}) S(f_1, \ldots, f_m, x_1, \ldots, x_n)$$

where Q_i is either $\forall$ or $\exists$ for $1 \leq i \leq m + n - l$, and where $s_1, \ldots, s_{m+n-l}$ are $f_1, \ldots, f_m, x_{l+1}, \ldots, x_n$ in some order.

$(Q_1 s_1)(Q_2 s_2) \ldots (Q_{m+n-l} s_{m+n-l}) S(f_1, \ldots, f_m, x_1, \ldots, x_n)$ is called a predicate form for R.
The reduced prefix is the sequence of quantifiers obtained by suppressing the quantifiers of type 0 from the prefix.

The Analytical Hierarchy

For $n > 0$, a Σ^1_n-prefix is one whose reduced prefix begins with $\exists^1$ and has $n-1$ alternations of quantifiers. For $n > 0$, a Π^1_n-prefix is one whose reduced prefix begins with $\forall^1$ and has $n-1$ alternations of quantifiers.

A Π^1_0-prefix or Σ^1_0-prefix is one whose reduced prefix is empty.

A predicate form is a Σ^1_n (Π^1_n)-form if it has a Σ^1_n (Π^1_n)-prefix.

The class of sets in $\mathbb{N}^l$ which can be expressed in Σ^1_n-form (respectively, Π^1_n-form) is denoted by Σ^1_n (respectively, Π^1_n).

The class $\Sigma^1_0 = \Pi^1_0$ is the class of arithmetical sets.

The Analytical Hierarchy

Theorem

For each integer $n \geq 1$,

(a) $\Sigma^1_n \cup \Pi^1_n \subsetneq \Sigma^1_{n+1} \cap \Pi^1_{n+1}$.

(b) *A set $R \subseteq \mathbb{N}^l$ is in the class Σ^1_n iff its complement is in the class Π^1_n.*

(c) $\Sigma^1_n - \Pi^1_n \neq \emptyset$ *and* $\Pi^1_n - \Sigma^1_n \neq \emptyset$.

Complete Sets

Definition

Given two sets $A, B \subseteq \mathbb{N}$ we say A is 1-reducible to B and write $A \leq_1 B$ if there exists a total computable injective function f from $\mathbb{N}$ to $\mathbb{N}$ such that $A = f^{-1}[B]$.

Definition

A set $A \subseteq \mathbb{N}$ is said to be Σ^1_n-complete (respectively, Π^1_n-complete) iff A is a Σ^1_n-set (respectively, Π^1_n-set) and for each Σ^1_n-set (respectively, Π^1_n-set) $B \subseteq \mathbb{N}$ it holds that $B \leq_1 A$.

Decision Problems

We denote $\mathcal{T}_z$ the non deterministic tiling system of index z, (accepting pictures over $\Sigma = \{a, b\}$), equipped with a Büchi acceptance condition.

Theorem

The non-emptiness problem and the infiniteness problem for Büchi-recognizable languages of infinite pictures are Σ^1_1-complete, i.e. :

1. $\{z \in \mathbb{N} \mid L(\mathcal{T}_z) \neq \emptyset\}$ *is* Σ^1_1*-complete.*
2. $\{z \in \mathbb{N} \mid L(\mathcal{T}_z)$ *is infinite* $\}$ *is* Σ^1_1*-complete.*

Proof. We express first $L(\mathcal{T}_z) \neq \emptyset$ by a Σ^1_1-formula:

$$\exists p \in \Sigma^{\omega,\omega} \ \ \exists \rho \in Q^{\omega,\omega} \ \ [\rho \text{ is a Büchi-accepting run of } \mathcal{T}_z \text{ on } p]$$

where "$[\rho$ is a Büchi-accepting run of $\mathcal{T}_z$ on $p]$" can be expressed by an arithmetical formula.

Decision Problems

Theorem

The universality problem for Büchi-recognizable languages of infinite pictures is Π^1_2-complete, i.e. : $\{z \in \mathbb{N} \mid L(\mathcal{T}_z) = \Sigma^{\omega,\omega}\}$ *is Π^1_2-complete.*

Proof. We express first $L(\mathcal{T}_z) = \Sigma^{\omega,\omega}$ by a Π^1_2-formula:

$$\forall p \in \Sigma^{\omega,\omega} \ \ \exists \rho \in Q^{\omega,\omega} \ \ [\rho \text{ is a Büchi-accepting run of } \mathcal{T}_z \text{ on } p]$$

Then the completeness result follows from the Π^1_2-completeness of the universality problem for ω-languages of Turing machines proved by Castro and Cucker (1989).

Decision Problems

Theorem

The inclusion and the equivalence problems for Büchi-recognizable languages of infinite pictures are Π^1_2-complete, i.e. :

1. $\{(y,z) \in \mathbb{N}^2 \mid L(\mathcal{T}_y) \subseteq L(\mathcal{T}_z)\}$ *is* Π^1_2*-complete.*
2. $\{(y,z) \in \mathbb{N}^2 \mid L(\mathcal{T}_y) = L(\mathcal{T}_z)\}$ *is* Π^1_2*-complete.*

We first express these sets by Π^1_2-formulas. The completeness result is deduced from corresponding results for Turing machines proved by Castro and Cucker (1989).

Unambiguity Problem

Theorem

The unambiguity problem for Büchi-recognizable languages of infinite pictures is Π^1_2-complete, i.e. :

$\{z \in \mathbb{N} \mid L(\mathcal{T}_z)$ is Büchi-recognizable by a unambiguous tiling system$\}$ is Π^1_2-complete.

Proof. We first express by a Π^1_2-formula that:

$L(\mathcal{T}_z)$ is Büchi-recognizable by a unambiguous tiling system.

Sketch of the Proof : A Dichotomy Result

Using a Turing machine of index z_0 such that $L(\mathcal{M}_{z_0})$ is not Borel, we define a reduction $H \circ \theta$ which is an injective computable function from $\mathbb{N}$ into $\mathbb{N}$ such that there are two cases.

First case. $L(\mathcal{M}_z) = \Sigma^\omega$. Then $L(\mathcal{T}_{H\circ\theta(z)}) = \Sigma^{\omega,\omega}$. In particular $L(\mathcal{T}_{H\circ\theta(z)})$ is unambiguous.

Second case. $L(\mathcal{M}_z) \neq \Sigma^\omega$. And $L(\mathcal{T}_{H\circ\theta(z)})$ is not a Borel set. But every unambiguous language of ω-pictures is Borel. Thus in that case the ω-picture language $L(\mathcal{T}_{H\circ\theta(z)})$ is inherently ambiguous.

Completeness Result

Finally, using the reduction $H \circ \theta$, we proved that :

$$\{z \in \mathbb{N} \mid L(\mathcal{M}_z) = \Sigma^\omega\} \leq_1 \{z \in \mathbb{N} \mid L(\mathcal{T}_z) \text{ is unambiguous } \}$$

And the completeness result follows from the Π^1_2-completeness of the universality problem for ω-languages of Turing machines.

Notice this implies also:

Theorem

$\{z \in \mathbb{N} \mid L(\mathcal{T}_z)$ *is a Borel set* $\}$ *is* Π^1_2*-hard.*

Determinizability and Complementability Problems

Theorem

The determinizability problem and the complementability problem for Büchi-recognizable languages of infinite pictures are Π^1_2-complete, i.e. :

1. $\{z \in \mathbb{N} \mid L(\mathcal{T}_z)$ *is Büchi-recognizable by a deterministic tiling system*$\}$ *is* Π^1_2*-complete.*
2. $\{z \in \mathbb{N} \mid \exists y \ \ \Sigma^{\omega,\omega} - L(\mathcal{T}_z) = L(\mathcal{T}_y)\}$ *is* Π^1_2*-complete.*

Proof. follows from the same dichotomy argument and topological arguments.

Cardinality Problems

A Büchi-recognizable language of infinite pictures is an analytic set. Thus it is either countable or has the cardinality of the continuum.

Theorem

1. $\{z \in \mathbb{N} \mid L(\mathcal{T}_z)$ *is countably infinite*$\}$ *is* $D_2(\Sigma^1_1)$*-complete.*
2. $\{z \in \mathbb{N} \mid L(\mathcal{T}_z)$ *is uncountable*$\}$ *is* Σ^1_1*-complete.*

What about the complement of a Büchi-recognizable language of infinite pictures ?

Set Theory

The usual axiomatic system **ZFC** is Zermelo-Fraenkel system **ZF** plus the axiom of choice **AC**. A model $(\mathbf{V}, \in)$ of the axiomatic system **ZFC** is a collection **V** of sets, equipped with the membership relation $\in$, where "$x \in y$" means that the set x is an element of the set y, which satisfies the axioms of **ZFC**.

The infinite cardinals are usually denoted by

$$\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\alpha, \ldots$$

The continuum hypothesis **CH** says $\aleph_1 = 2^{\aleph_0}$ where $2^{\aleph_0}$ is the cardinal of the continuum.

Theorem

The cardinality of the complement of a Büchi-recognizable language of infinite pictures is not determined by the axiomatic system **ZFC**. *Indeed there is a Büchi tiling system* $\mathcal{T}$ *such that:*

1. *There is a model* V_1 *of* **ZFC** *in which* $\{a,b\}^{\omega,\omega} - L(\mathcal{T})$ *is countable.*
2. *There is a model* V_2 *of* **ZFC** *in which* $\{a,b\}^{\omega,\omega} - L(\mathcal{T})$ *has cardinal* $2^{\aleph_0}$.
3. *There is a model* V_3 *of* **ZFC** *in which* $\{a,b\}^{\omega,\omega} - L(\mathcal{T})$ *has cardinal* $\aleph_1$ *with* $\aleph_0 < \aleph_1 < 2^{\aleph_0}$.

Cardinality problems

Using Shoenfield's Theorem we show that some of these problems are located at the third level of the analytical hierarchy.

Theorem

1. $\{z \in \mathbb{N} \mid \{a,b\}^{\omega,\omega} - L(\mathcal{T}_z)$ *is finite* $\}$ *is* Π^1_2*-complete.*
2. $\{z \in \mathbb{N} \mid \{a,b\}^{\omega,\omega} - L(\mathcal{T}_z)$ *is countable* $\}$ *is* $\Sigma^1_3 \setminus (\Pi^1_2 \cup \Sigma^1_2)$.
3. $\{z \in \mathbb{N} \mid \{a,b\}^{\omega,\omega} - L(\mathcal{T}_z)$ *is uncountable*$\}$ *is* $\Pi^1_3 \setminus (\Pi^1_2 \cup \Sigma^1_2)$.

Theorem

The topological complexity of a Büchi-recognizable language of infinite pictures is not determined by the axiomatic system **ZFC**. *Indeed there is a Büchi tiling system* $\mathcal{T}$ *such that:*

1. *There is a model* V_1 *of* **ZFC** *in which the* ω*-picture language* $L(\mathcal{T})$ *is an analytic but non Borel set.*
2. *There is a model* V_2 *of* **ZFC** *in which the* ω*-picture language* $L(\mathcal{T})$ *is a Borel* Π^0_2*-set.*

From the proof of this result we can infer the following one:

Theorem

$\{z \in \mathbb{N} \mid L(\mathcal{T}_z)$ *is Borel* $\}$ *is not in* $(\Pi^1_2 \cup \Sigma^1_2)$.

Similar results for finite machines

Surprisingly we have the following result:

Theorem

The topological complexity of an ω-language accepted by a one-counter Büchi automaton or by a 2-tape Büchi automaton is not determined by the axiomatic system **ZFC**. *Indeed there is a one-counter Büchi automaton $\mathcal{A}$ and a 2-tape Büchi automaton $\mathcal{B}$ such that:*

1. *There is a model V_1 of* **ZFC** *in which the ω-languages $L(\mathcal{A})$ and $L(\mathcal{B})$ are analytic but non Borel sets.*
2. *There is a model V_2 of* **ZFC** *in which the ω-languages $L(\mathcal{A})$ and $L(\mathcal{B})$ are Borel Π^0_2-sets.*

The ordinal γ^1_2 may depend on set theoretic axioms

The ordinal γ^1_2 is the least basis for subsets of ω_1 which are Π^1_2 in the codes.

It is the least ordinal such that whenever $X \subseteq \omega_1$, $X \neq \emptyset$, and $\hat{X} \subseteq WO$ is Π^1_2, there is $\beta \in X$ such that $\beta < \gamma^1_2$.

The least ordinal which is not a Δ^1_n-ordinal is denoted δ^1_n.

Theorem (Kechris, Marker and Sami 1989)

- (ZFC) $\delta^1_2 < \gamma^1_2$
- (V = L) $\gamma^1_2 = \delta^1_3$
- (Π^1_1-Determinacy) $\gamma^1_2 < \delta^1_3$

Are there effective analytic sets of every Borel rank $\alpha < \gamma^1_2$?

Complexity of ω-languages of deterministic machines

deterministic finite automata (Landweber 1969)

- **ω-regular languages** accepted by **deterministic Büchi automata** are Π^0_2-sets.
- **ω-regular languages** are boolean combinations of Π^0_2-sets hence Δ^0_3-sets.

deterministic Turing machines

- **ω-languages** accepted by **deterministic Büchi Turing machines** are Π^0_2-sets.
- **ω-languages** accepted by **deterministic Muller Turing machines** are boolean combinations of Π^0_2-sets hence Δ^0_3-sets.

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