A model–theoretic approach to parameter free Π_2–induction

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Introduction

- **General Goal**: To obtain nice and informative characterizations of $Th_\Gamma(\mathbf{I}\Pi_n^-)$, for $n = 1, 2$ and $\Gamma = \Pi_n, \mathcal{B}(\Sigma_n)$.
- Nice: We look for simple and clean characterizations. If possible, they should have the form of an *induction axiom or rule*. Alternatively, other related principles can be used (for instance, iteration,...).
- Informative: We look for non trivial characterizations that shed light on interesting questions on the fragments at hands:
  - The class of provably total computable functions
  - Relative strength (in comparison with other fragments)
  - Relations with other formulations of the induction principle (for instance, by means of inference rules). Etc.

Two conservation results

Theorem (Kaye–Paris–Dimitracopoulos,'88)

1. $\mathbf{I}\Pi_1^-$ *is* Π_2 *conservative over* $\mathbf{I}\Delta_0 + \exp$.
2. $\mathbf{I}\Delta_0 + \exp$ *is* Σ_2 *conservative over* $\mathbf{I}\Pi_1^-$.

Theorem (Beklemishev, '99)

$\mathbf{I}\Pi_2^-$ *is* $\mathcal{B}(\Sigma_2)$ *conservative over* $\mathbf{I}\Sigma_1^-$.

Corollary

1. $Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Pi_1^-) \equiv Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Delta_0 + \exp)$
2. $Th_{\mathcal{B}(\Sigma_2)}(\mathbf{I}\Pi_2^-) \equiv \mathbf{I}\Sigma_1^-$

KPD's results on $\mathbf{I}\Pi_1^-$

- Both results have model–theoretic proofs.
- Both proofs deal with *ad hoc* model theoretic constructions.
- The characterization of $Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Pi_1^-)$ is informative, but it is not very "nice".
- The theory $\mathbf{I}\Delta_0 + \exp$ is too strong. A finer result should be possible.
- These results have generalizations to conservation results for $\mathbf{I}\Sigma_1 + \mathbf{I}\Pi_2^-$.
- Similar results for $\mathbf{I}\Pi_2^-$ are unknown.

Beklemishev's result on $\mathsf{I}\Pi_2^-$

It is a good example of a nice and informative characterization.

- $Th_{\mathcal{B}(\Sigma_2)}(\mathsf{I}\Pi_2^-)$ is described in terms of parameter free Σ_1–induction.
- It gives an *indirect* characterization of the class of provably total computable functions of $\mathsf{I}\Pi_2^-$: it is the class of the primitive recursive functions.
  - This answer a question posed by R. Kaye.
- This conservation result is obtained as a corollary of a general conservation result between *reflection principles*.
  - First, parameter free Π_2 and Σ_1–induction are proved to be equivalent to certain reflection principles.
  - Then, the conservation result between these reflection principles is derived using some results from **Provability Logic**.
- This result generalizes to $\mathsf{I}\Pi_n^-$, for $n \geq 1$; but, there is no corresponding result for $\mathsf{I}\Pi_1^-$

Goals

- To develop *uniform* proofs of KPD's and Beklemishev results.
- To bring to light more explicit relations between both results.
- To give a direct argument showing that the provably total computable functions of $\mathsf{I}\Pi_2^-$ coincide with the primitive recursive ones.

Outline

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- Beklemishev's result on $\mathrm{I}\Pi_2^-$
- Goals

Background
- Existentially closed models
- Parsons's theorem

An analysis of Π_n–IR_0
- Induction up to Σ_n–definable elements
- The rule Π_n^-–IR_0

Σ_{n+1}–closed models

- Σ_{n+1}–closed models provide a simple and clear method to obtain conservation results between theories described by *axiom schemes* and *inference rules* versions of them.
- **Definition**. Let T be a theory. We say that $\mathfrak{A} \models T$ is a Σ_{n+1}–closed model of T if for each $\mathfrak{B} \models T$,

$$\mathfrak{A} \prec_n \mathfrak{B} \quad \Longrightarrow \quad \mathfrak{A} \prec_{n+1} \mathfrak{B}$$

- It generalizes the notion of an *existentially closed model*.
- **Proposition**. (Existence)
  Let T be a Π_{n+2}–axiomatizable theory and $\mathfrak{A} \models \mathbf{T}$ countable. Then there exists $\mathfrak{B} \models T$ such that $\mathfrak{A} \prec_n \mathfrak{B}$ and $\mathfrak{B}$ is Σ_{n+1}–closed for T.
- **Corollary**. Every consistent and Π_{n+2}–axiomatizable theory has a Σ_{n+1}–closed model.

The basics of the method

- The method is very general. It leans upon the syntactical structure of the axiom scheme and no *ad hoc* model–theoretic construction is involved.
- The basic device is the following result:

Theorem (Avigad,'02)

Let T_1 be a Π_{n+2}–axiomatizable theory such that every Σ_{n+1}–closed model for T_1 is a model of $\mathbf{T}_2$. Then T_2 is Π_{n+1}–conservative over T_1.

The basics of the method (II)

- A typical application includes a reduction between axioms and inference rules that usually is carried out using next lemma:

Lemma

Let $\mathfrak{A}$ be a Σ_{n+1}–closed model for T. Let $\varphi(\vec{x}) \in \Pi_{n+1}$ and $\vec{a} \in \mathfrak{A}$ such that $\mathfrak{A} \models \varphi(\vec{a})$. Then there exist $\theta(v, \vec{x}) \in \Pi_n$ and $b \in \mathfrak{A}$ such that

$$\mathfrak{A} \models \theta(b, \vec{a}) \quad \textit{and} \quad T \vdash \theta(v, \vec{x}) \rightarrow \varphi(\vec{x})$$

Parsons's Theorem (I)

A typical application of the method gives us a simple proof of the following theorem of Parsons:

Theorem (Parsons, '72)

$\mathbf{I}\Sigma_1$ *is* Π_2*–conservative over* $\mathbf{I}\Delta_0 + \Sigma_1$*–IR.*

- Induction scheme:

$$\mathbf{I}_\varphi : \qquad \varphi(0) \wedge \forall x\,(\varphi(x) \to \varphi(x+1)) \to \forall x\,\varphi(x)$$

- Induction rule (IR):

$$\frac{\varphi(0), \quad \forall x\,(\varphi(x) \to \varphi(x+1))}{\forall x\,\varphi(x)}$$

- For each theory T, $T + \Sigma_1$–IR denotes the closure of T under first order logic and (nested) applications of Σ_1–IR.
- $[T; \Sigma_1\text{–IR}]$ denotes the closure of T under first order logic and unnested applications of Σ_1–IR.

Parsons' Theorem (II)

Theorem

Let T be a theory and $\mathfrak{A}$ be a Σ_2–closed model for $T + \Sigma_1$–IR. Then $\mathfrak{A} \models \mathbf{I}\Sigma_1$.

Corollary

Let T be a Π_3–axiomatizable theory. Then $T + \mathbf{I}\Sigma_1$ is Π_2–conservative over $T + \Sigma_1$–IR.

- For $T = \mathbf{I}\Delta_0$ we obtain Parsons's Theorem.

Induction up to Σ_{n+1}–definable elements

- If Γ is a set of formulas and $n \geq 0$, we denote by $\mathbf{I}(\Gamma, \mathcal{K}_{n+1})$ the theory given by $\mathbf{I}\Sigma_n^-$ together with the induction scheme

$$\varphi(0) \wedge \forall x\,(\varphi(x) \to \varphi(x+1)) \to \\ \forall x_1, x_2\,(\delta(x_1) \wedge \delta(x_2) \to x_1 = x_2) \to \forall x\,(\delta(x) \to \varphi(x))$$

  where $\varphi(x) \in \Gamma$ and $\delta(x) \in \Sigma_{n+1}^-$.
- We consider two inference rules associated with this scheme:
  - $(\Gamma, \mathcal{K}_{n+1})$–IR denotes the following inference rule:

$$\frac{\varphi(0) \wedge \forall x\,(\varphi(x) \to \varphi(x+1))}{\forall x_1, x_2\,(\delta(x_1) \wedge \delta(x_2) \to x_1 = x_2) \to \forall x\,(\delta(x) \to \varphi(x))}$$

  - $(\Gamma, \mathcal{K}_{n+1})$–$\mathrm{IR}_0$ denotes the rule:

$$\frac{\forall x\,(\varphi(x) \to \varphi(x+1))}{\forall x_1, x_2\,(\delta(x_1) \wedge \delta(x_2) \to x_1 = x_2) \to \varphi(0) \to \forall x\,(\delta(x) \to \varphi(x))}$$

  where, in both cases, $\delta(x) \in \Sigma_{n+1}^-$ and $\varphi(x) \in \Gamma$.

Some basic facts on $\mathbf{I}(\Sigma_1, \mathcal{K}_1)$

- A version of Parsons's theorem holds for Σ_1–induction up to Σ_1–definable elements.

Lemma

Let T be an Π_2-axiomatizable extension of $I\Delta_0$. Then $T + \mathbf{I}(\Sigma_1, \mathcal{K}_1)$ is Π_2–conservative over $T + (\Sigma_1, \mathcal{K}_1)$–IR.

- The parameter free version of $\mathbf{I}(\Sigma_1, \mathcal{K}_1)$ is equivalent to $\mathbf{I}\Pi_1^-$.
- So, $\mathbf{I}\Delta_0 + (\Sigma_1, \mathcal{K}_1)$–IR is provides an upper bound for the class of Π_2–consequences of $\mathbf{I}\Pi_1^-$.

A proof of KPD's results

Lemma

The following theories are equivalent:

1. $\mathbf{I}\Delta_0 + (\Sigma_1, \mathcal{K}_1)$–*IR*.
2. $[\mathbf{I}\Delta_0, (\Sigma_1, \mathcal{K}_1)$–*IR*$]$.

Theorem

1. $\mathbf{I}\Pi_1^-$ *is* Π_2–*conservative over* $\mathbf{I}\Delta_0 + \exp$.
2. $Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Pi_1^-) \equiv [\mathbf{I}\Delta_0, (\Sigma_1^-, \mathcal{K}_1)$–*IR*$]$.

A proof of KPD's results (II)

Proof:

1. Let $\theta \in \Pi_2$ such that $\mathbf{I}\Pi_1^- \vdash \theta$. Then, $\mathbf{I}\Delta_0 + (\Sigma_1, \mathcal{K}_1)$–IR $\vdash \theta$. By previous Lemma, $[\mathbf{I}\Delta_0, (\Sigma_1, \mathcal{K}_1)$–IR$] \vdash \theta$. Hence, $\mathbf{I}\Delta_0 + \exp \vdash \theta$, since $\mathbf{I}\Delta_0 + \exp$ is equivalent to $[\mathbf{I}\Delta_0, \Sigma_1$–IR$]$.

2. It is enough to prove that $\mathbf{I}\Pi_1^-$ is $\mathcal{B}(\Sigma_1)$-conservative over $[\mathbf{I}\Delta_0, (\Sigma_1^-, \mathcal{K}_1)$–IR$]$.

   Let $\theta \in \mathcal{B}(\Sigma_1)$ such that $\mathbf{I}\Pi_1^- \vdash \theta$. Then $\theta \in \Pi_2$ and thus, $[\mathbf{I}\Delta_0, (\Sigma_1, \mathcal{K}_1)$–IR$] \vdash \theta$. Now let us observe that in each model of $\mathfrak{A} \models [\mathbf{I}\Delta_0, (\Sigma_1, \mathcal{K}_1)$–IR$]$, $\mathcal{K}_1(\mathfrak{A}) \prec_1 \mathfrak{A}$ and $\mathcal{K}_1(\mathfrak{A}) \models [\mathbf{I}\Delta_0, (\Sigma_1^-, \mathcal{K}_1)$–IR$]$. So, $[\mathbf{I}\Delta_0, (\Sigma_1, \mathcal{K}_1)$–IR$]$ is Σ_2–conservative over $[\mathbf{I}\Delta_0, (\Sigma_1^-, \mathcal{K}_1)$–IR$]$. Since $\theta \in \mathcal{B}(\Sigma_1) \subseteq \Sigma_2$ the result follows.

The rules Π_n^-–IR_0 and Π_n–IR

Let T be an extension of $\mathbf{I}\Delta_0$. Then

- $[T, \Pi_1\text{–}\mathrm{IR}_0] \equiv [T, \Sigma_1\text{–}\mathrm{IR}] \equiv [T, \Sigma_1\text{–}\mathrm{IR}_0]$
- For parameter free rules:
  - $[T, \Pi_1\text{–}\mathrm{IR}] \equiv [T, \Pi_1^-\text{–}\mathrm{IR}]$
  - $[T, \Sigma_1\text{–}\mathrm{IR}] \equiv [T, \Sigma_1^-\text{–}\mathrm{IR}]$
- Rules Π_n–IR_0 and Π_n^-–IR_0 are not equivalent.

Theorem (Beklemishev,'97)

If T is Π_2–axiomatizable, then

$$[T, \Sigma_1\text{–}IR] \equiv [T, \Pi_2\text{–}IR]$$

Analyzing Π_1^-–IR_0

Lemma

Let T be a Π_2–axiomatizable extension of $\mathbf{I}\Delta_0$. Then, the following theories are equivalent:

1. $[T, \Pi_1^-\text{–}IR_0]$.
2. $[T, (\Sigma_1^-, \mathcal{K}_1)\text{–}IR]$.
3. $[T, (\Sigma_1^-, \mathcal{K}_1)\text{–}IR_0]$.
4. $[T, (\Pi_1^-, \mathcal{K}_1)\text{–}IR_0]$.

Proposition

Let T be a Π_2–axiomatizable extension of $\mathbf{I}\Delta_0$. Then $T + \Pi_1^-\text{–}IR_0$ is Π_1–conservative over $T + \Pi_1\text{–}IR$.

Characterizing $Th_{\Pi_1}(\mathbf{I}\Pi_1^-)$ and $Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Pi_1^-)$

Theorem

1. $[\mathbf{I}\Delta_0, \Pi_1\text{–}IR_0]$ *is* Σ_2*–conservative over* $[\mathbf{I}\Delta_0, \Pi_1^-\text{–}IR_0]$. *So,* $\mathbf{I}\Delta_0 + \exp$ *(*$\equiv [\mathbf{I}\Delta_0, \Pi_1\text{–}IR_0]$*) is* Σ_2*–conservative over* $\mathbf{I}\Pi_1^-$.
2. $Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Pi_1^-) \equiv \mathbf{I}\Delta_0 + \Pi_1^-\text{–}IR_0 \equiv [\mathbf{I}\Delta_0, \Pi_1^-\text{–}IR_0]$.
3. $Th_{\Pi_1}(\mathbf{I}\Pi_1^-) \equiv \mathbf{I}\Delta_0 + \Pi_1\text{–}IR$.

Analyzing Π_2^-–IR_0

For Π_2^-–IR_0 we can obtain a specific result relating this rule with $\mathbf{I}\Sigma_1^-$.

Theorem

If T is a $\mathcal{B}(\Sigma_2)$–axiomatizable extension of $\mathbf{I}\Delta_0$, then $[T, \Pi_2^-\text{–}IR_0]$ and $T + \mathbf{I}\Sigma_1^-$ are equivalent.

Corollary

1. *If T is a $\mathcal{B}(\Sigma_2)$–axiomatizable extension of $\mathbf{I}\Sigma_0$ then $T + \Pi_2^-\text{–}IR_0 \equiv [T, \Pi_2^-\text{–}IR_0]$.*
2. *$\mathbf{I}\Sigma_1^-$ is closed under Π_2^-–IR_0 and Π_2–IR (but $\mathbf{I}\Sigma_1$ is not).*

A proof of Beklemishev's result

Proposition

Let T be a $\mathcal{B}(\Sigma_2)$–axiomatizable theory closed under Π_2^-–IR_0. Then $T + (\Sigma_2, \mathcal{K}_2)$–$IR$ is a Π_2–conservative extension of T.

Theorem

1. *If T is a $\mathcal{B}(\Sigma_2)$–axiomatizable theory, then $T + \mathbf{I}\Pi_2^-$ is $\mathcal{B}(\Sigma_2)$–conservative over $T + \Pi_2^-$–IR_0.*
2. *(Beklemishev) $\mathbf{I}\Pi_2^-$ is $\mathcal{B}(\Sigma_2)$–conservative over $\mathbf{I}\Sigma_1^-$.*

A proof of Beklemishev's result (II)

Proof:

(1) Let $\theta \in \mathcal{B}(\Sigma_2)$ such that $T + \mathbf{I}\Pi_2^- \vdash \theta$. Then, using conjunctive normal form, θ is equivalent to a conjunction of sentences of the form $\psi_1 \to \psi_2$, where ψ_1, $\psi_2 \in \Pi_2$; hence, we can assume without lost of generality that θ is $\psi_1 \to \psi_2$.
Then $(T + \psi_1) + \mathbf{I}\Pi_2^- \vdash \psi_2$ implies $(T + \psi_1) + (\Sigma_2, \mathcal{K}_2)$–$\mathbf{IR} \vdash \psi_2$.
By previous proposition, $(T + \psi_1) + \Pi_2^-$–$\mathbf{IR}_0 \vdash \psi_2$ and we can conclude that $T + \Pi_2^-$–$\mathbf{IR}_0 \vdash \psi_1 \to \psi_2$, since it can be easily checked that $(T + \psi_1) + \Pi_2^-$–$\mathbf{IR}_0 \equiv (T + \Pi_2^-$–$\mathbf{IR}_0) + \psi_1$.

(2) Let us recall that $\mathbf{I}\Sigma_1^-$ is $\mathcal{B}(\Sigma_1)$–axiomatizable and it is also closed under Π_2^-–$\mathbf{IR}_0$; hence, part (2) follows from part (1), applied to $T = \mathbf{I}\Sigma_1^-$.

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