On a problem of J. Paris

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   - Early results
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Some axiom schemas

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- $\mathbf{I\Delta_n}$: induction schema for Δ_n formulae (plus PA^-)

$$\forall x(\varphi(x) \leftrightarrow \psi(x)) \rightarrow [\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(x')) \rightarrow \forall x \varphi(x)]$$

for $\varphi \in \Sigma_n$ and $\psi \in \Pi_n$

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Paris-Kirby 1978

J. Paris and L. Kirby: Σ_n-*collection schemas in arithmetic*, Logic Colloquium '77, North-Holland, Amsterdam, 199–209, 1978.

Early results

Theorem (Paris-Kirby 1978)

For all $n \in \mathbf{N}$, $I\Sigma_{n+1} \Rightarrow B\Sigma_{n+1} \Rightarrow I\Sigma_n \Leftrightarrow L\Sigma_n$
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Does $I\Delta_{n+1}$ imply $B\Sigma_{n+1}$ for all $n \in \mathbf{N}$?
Problem 34 in Clote-Krajicek's list of open problems.

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Slaman 2004

T. A. Slaman: *Σ_n-bounding and Δ_n-induction*, Proc. Amer. Math. Soc. 132 (2004), 2449–2456.

Slaman's result

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$\mathrm{I}\Delta_2 \Rightarrow I\Sigma_1 \Rightarrow exp$.

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For any $n > 0$, $\mathrm{I}\Delta_{n+1} \Rightarrow \mathrm{B}\Sigma_{n+1}$.

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Problem 2 (Beklemishev 2003)

Does $I\Delta_1 + \Pi_2(\mathbb{N})$ imply $B\Sigma_1$?

A picture

Answer to Problem 2

Slaman's Theorem implies "yes", since $\exp$ belongs to $\Pi_2(\mathbb{N})$.

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$$\begin{array}{ccccc} \Pi_2(\mathbb{N}) + \mathrm{I}\Delta_1 & \Rightarrow & \Pi_2(\mathbb{N}) & \Rightarrow & \mathrm{I}\Delta_0 + exp \\ \Downarrow & & \Downarrow\!\!\!\!\!\!\not\quad & & \\ \mathrm{B}\Sigma_1 & \Rightarrow & \mathrm{I}\Delta_1 & & \end{array}$$

Thapen's improvement

Thapen 2005

N. Thapen: *A note on* Δ_1 *induction*, Fund. Math. 186 (2005), 79–84.

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Theorem (Thapen 2005)

$I\Delta_1 + \Omega_{1,k} \Rightarrow \mathrm{B}\Sigma_1$, *for any* $k \geq 1$.

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Theorem (Thapen 2005)

$I\Delta_1 + th_p \Rightarrow \mathrm{B}\Sigma_1$,

where th_p *denotes the axiom* $\forall x \exists y \exists z\ (x{<}p(y) \wedge z = x^y)$, *with* p *any primitive recursive function.*

A new problem

Problem 1''

Does every nonstandard model of $I\Delta_1$ satisfying the property

$$\forall a{\in}M \exists \gamma{\in}M{-}\mathbb{N} \text{ such that } \; a^{\gamma} \in M$$

satisfy $B\Sigma_1$?

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Facts

- Thapen's axiom th_p implies the above property, for any p (since $\mathbb{N}$ is closed under all primitive recursive functions).
- The property above amounts to saying that $M \neq a^{\mathbb{N}}$ for any $a \in M$.

An early result

Theorem (Paris-Kirby 1978)

(i) For any countable $M \models I\Delta_0$ *and* $n \geq 2$,

$$M \models B\Sigma_n \Leftrightarrow \text{ there exists } K \models I\Delta_0 \text{ such that } M <_{n,e} K.$$

(ii) M *has a proper* Σ_1*-elementary end extension* $\Rightarrow M \models B\Sigma_2$ *(and hence there is* $K \models I\Delta_0$ *such that* $M <_{2,e} K$*).*

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Natural questions

(a) $M \models B\Sigma_1 \Leftrightarrow$ there exists $K \models I\Delta_0$ such that $M <_{1,e} K$?

(b) $M \models B\Sigma_0 \Leftrightarrow$ there exists $K \models I\Delta_0$ such that $M <_{0,e} K$?

Only one remaining problem

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- There exists countable $K \models \mathrm{B}\Sigma_1 + \neg\mathrm{B}\Sigma_2$.
- $B\Sigma_0 \iff B\Sigma_1$.
- $M \subset_e K \Rightarrow M <_0 K$ always.

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Wilkie-Paris 1989

A. Wilkie and J. Paris: *On the existence of end extensions of models of bounded induction*, in Logic, Methodology and Philosophy of Science VIII (Moscow 1987), 143–161, North-Holland, Amsterdam, 1989.

A variant and an observation

Problem 3 (J. Paris)

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Observation (Sirokofskich-D. 2007)

Either Problem 1' or Problem 3' **has** a negative solution.

Facts needed

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A. Wilkie and J. Paris: *On the scheme of induction for bounded arithmetic formulas*, Ann. Pure Appl. Logic 35 (1987), 261–302.

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$\mathrm{B}\Sigma_1$ is Π_2-conservative over $\mathrm{I}\Delta_0$.

Sketch of proof

Definition

A model M of a theory T is said to be ω_1-like if

- *card(M)=ω_1,*
- *card$\{b \in M : M \models b < a\} \leq \aleph_0$, for all $a \in M$.*

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• Then M_1 is a model of $\mathrm{B}\Sigma_1$, because of the positive solution of Problem 1'.

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- Hence there is a model M_2 of $\mathrm{I}\Delta_1$ such that $M_1 \subset_e M_2$.

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- But all ω_1-like models of $\mathrm{I}\Delta_0$ satisfy $\mathrm{B}\Sigma_n$, for all $n \in \mathbb{N}$.

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- But all ω_1-like models of $\mathrm{I}\Delta_0$ satisfy $\mathrm{B}\Sigma_n$, for all $n \in \mathbb{N}$.
- Hence M_{ω_1} is model of PA.
- Note that $M_{\omega_1} \models \exists x$ "x is the Gödel number of a proof of a contradiction from $\mathrm{I}\Delta_0$".

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- But all ω_1-like models of $\mathrm{I}\Delta_0$ satisfy $\mathrm{B}\Sigma_n$, for all $n \in \mathbb{N}$.
- Hence M_{ω_1} is model of PA.
- Note that $M_{\omega_1} \models \exists x$ "x is the Gödel number of a proof of a contradiction from $\mathrm{I}\Delta_0$".
- But this is impossible, since $PA \vdash Con(\mathrm{I}\Delta_0)$.