Parameter Free Π_1 Induction and Restricted Exponentiation

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Introduction

- General Question: to characterize the sets $Th_\Gamma(T)$ of all theorems of T of a specifical logical complexity Γ.

- We shall deal with this question for:

$$\begin{aligned} T &= \text{ scheme of parameter free } \Pi_1\text{–induction } \mathbf{I}\Pi_1^- \\ \Gamma &= \Pi_2,\ \mathcal{B}(\Sigma_1) \end{aligned}$$

- $\mathbf{I}\Pi_1^-$ is P^- together with the *induction scheme*

$$(I_\varphi)\quad \varphi(0) \wedge \forall x\,(\varphi(x) \to \varphi(x+1)) \ \to\ \forall x\,\varphi(x)$$

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Why $\mathbf{I}\Pi_1^-$?

Theorem (Bigorajska,'95)

Suppose $\varphi(x,y) \in \Sigma_1$ *and* $\mathbf{I}\Pi_1^- \vdash \forall x\, \exists y\, \varphi(x,y)$. *Then, there is a term* $t(x)$ *such that* $\mathbf{I}\Pi_1^- \vdash \exists u\, \forall x > u \exists y \leq t(x)\, \varphi(x,y)$.

- $\mathbf{I}\Pi_1^-$ cannot prove the totality of functions of more than polynomial growth: $R(\mathbf{I}\Pi_1^-) = R(\mathbf{I}\Delta_0) = \mathcal{M}^2$.

Theorem (Kaye–Paris–Dimitracopoulos,'88)

1. $\mathbf{I}\Delta_0 + \mathit{exp}$ *is conservative over* $\mathbf{I}\Pi_1^-$ *w.r.t.* Σ_2 *sentences.*
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A hierarchy between $\mathbf{I}\Delta_0$ and $\mathbf{I}\Delta_0+exp$

$$\mathbf{I}\Delta_0 \subset \begin{array}{c} Th_{\Pi_1}(\mathbf{I}\Pi_1^-) \\ = \\ Th_{\Pi_1}(\mathbf{I}\Delta_0+exp) \end{array} \subseteq \begin{array}{c} Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Pi_1^-) \\ = \\ Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Delta_0+exp) \end{array} \subseteq Th_{\Pi_2}(\mathbf{I}\Pi_1^-) \subset \mathbf{I}\Delta_0+exp$$

- Wilkie and Paris characterized the Π_1 theorems of $\mathbf{I}\Pi_1^-$ and proved that $\mathbf{I}\Delta_0 \nvdash Th_{\Pi_1}(\mathbf{I}\Pi_1^-)$.
- No similar results were known for the $\mathcal{B}(\Sigma_1)$ and the Π_2 theorems of $\mathbf{I}\Pi_1^-$.
- We do not know *good candidates* to separate $Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Pi_1^-)$ and $Th_{\Pi_2}(\mathbf{I}\Pi_1^-)$.

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Our results

- We characterize $Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Pi_1^-)$ and $Th_{\Pi_2}(\mathbf{I}\Pi_1^-)$ in terms of exponentiation restricted to Σ_1–definable elements.
- We prove that $Th_{\Pi_1}(\mathbf{I}\Pi_1^-) \subset Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Pi_1^-) \subset Th_{\Pi_2}(\mathbf{I}\Pi_1^-)$ form a proper hierarchy.

Our methods

- Model–theoretic constructions based on *Kaye–Paris–Dimitracopoulos: On parameter free induction schemas. J. Symbolic Logic'88.*
- We make explicit that $\mathbf{I}\Pi_1^-$ allows for restricted Σ_1 induction: "induction up to Σ_1–definable elements."

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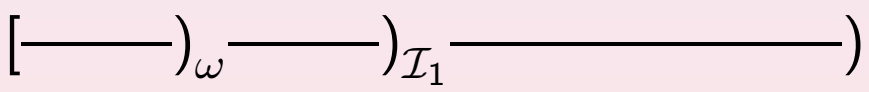
Definable and minimal elements

- a is Γ–definable in $\mathfrak{A}$ (with parameters $\bar{b}$) if there is $\varphi(x, \bar{v}) \in \Gamma$ such that $\mathfrak{A} \models \varphi(a, \bar{b}) \wedge \exists! x\, \varphi(x, \bar{b})$.

- a is Γ–minimal in $\mathfrak{A}$ (with parameters $\bar{b}$) if there is $\varphi(x, \bar{v}) \in \Gamma$ such that $\mathfrak{A} \models a = (\mu x)\,(\varphi(x, \bar{b}))$.

- $\mathcal{K}_1(\mathfrak{A}) = \Sigma_1$–definable elements of $\mathfrak{A}$.

- $\mathcal{I}_1(\mathfrak{A}) =$ initial segment determined by $\mathcal{K}_1(\mathfrak{A})$

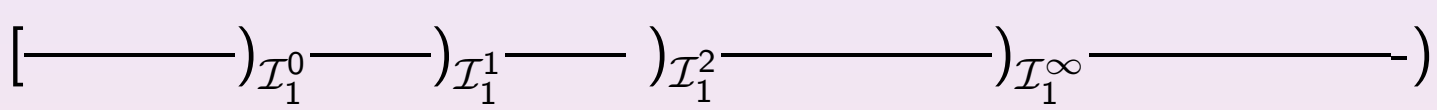


Iterating Σ_1–definability: $\mathcal{I}_1^\infty$

$$\mathcal{I}_1^0(\mathfrak{A}) = \mathcal{I}_1(\mathfrak{A})$$

$$\mathcal{I}_1^{k+1}(\mathfrak{A}) = \text{initial segment determined by } \mathcal{K}_1(\mathfrak{A}, \mathcal{I}_1^k(\mathfrak{A}))$$

$$\mathcal{I}_1^\infty(\mathfrak{A}) = \bigcup_{k \geq 0} \mathcal{I}_1^k(\mathfrak{A})$$



- $\mathcal{I}_1^\infty(\mathfrak{A})$ is the least initial segment of $\mathfrak{A}$ containing all the Σ_1–definable elements and closed under Σ_1–definability.
- If $\mathfrak{A} \models \mathbf{I}\Sigma_1$ with nonstandard Σ_1–definable elements, $\{\mathcal{I}_1^k(\mathfrak{A}) : k \geq 0\}$ form a proper hierarchy.

$\mathsf{I}\Pi_1^-$ and Restricted Σ_1–induction

Proposition

Over $\mathsf{I}\Delta_0$ *the following are equivalent:*

1. $\mathfrak{A} \models \mathsf{I}\Pi_1^-$.
2. *For each* $\varphi(x, v) \in \Sigma_1$ *and* $a, b \in \mathcal{I}_1(\mathfrak{A})$*:*
$$\mathfrak{A} \models \varphi(0, b) \wedge \forall x\, (\varphi(x, b) \to \varphi(x+1, b)) \to \varphi(a, b).$$

Proof:

Pick $c \in \mathcal{K}_1(\mathfrak{A})$ with $a, b \leq c$.

Step 1: $\mathsf{I}\Pi_1^- \vdash$ minimization for Π_1 formulas "witnessed" in $\mathcal{K}_1(\mathfrak{A})$.

Step 2: Show that $\varphi(x, b) \equiv x \in d$, for all $x \leq c$.

Step 3: Apply Δ_0–induction to $x \in d$. □

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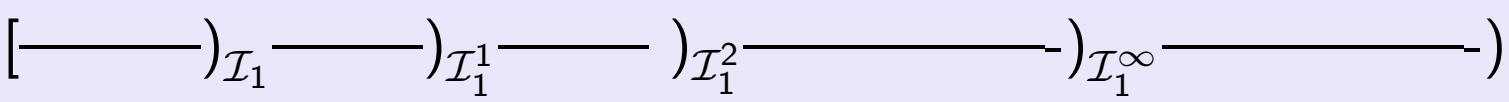
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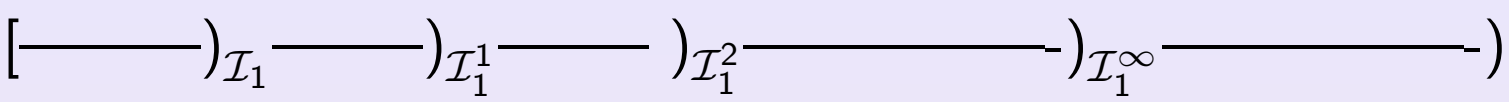
Pick $c \in \mathcal{K}_1(\mathfrak{A}, \mathcal{I}_1(\mathfrak{A}))$ with $b \leq c$. If $\delta(u, d) \in \Sigma_1$ defines c with $d \in \mathcal{I}_1(\mathfrak{A})$, apply Σ_1–induction up to a to:

$$\exists y\,(y = c^x) \ \equiv\ \exists y, u\,(\delta(u, d) \wedge y = u^x)$$

Then, c^a exists and so does b^a since $b \leq c$. □

- Previous theorem cannot be extended. It captures the *exact amount* of exponentiation available in models of $\mathbf{I}\Pi_1^-$.

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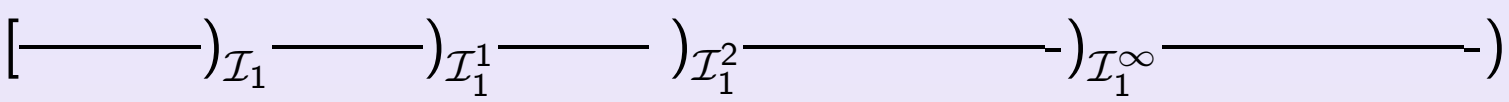
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Expressing "$\forall x \in \mathcal{I}_1(\mathfrak{A})$" in the language of Arithmetic

- For each $a \in \mathcal{K}_1(\mathfrak{A})$ there is b Π_0–minimal such that $a = (b)_0$.

$$\text{“}\forall x \in \mathcal{I}_1(\mathfrak{A})\, \Phi(x, \bar{v})\text{”}$$

$$\Updownarrow$$

$$\{\forall z, x \left(\left\{ \begin{array}{rl} z = & (\mu t)\,(\delta(t)) \\ \wedge & x \leq (z)_0 \end{array} \right\} \rightarrow \Phi(x, \bar{v}) \right) : \delta(t) \in \Pi_0^- \}$$

- “$\forall a \in \mathcal{I}_1(\mathfrak{A})\ \forall b \in \mathcal{I}_1^1(\mathfrak{A})\ \exists y\, (y = b^a)$” can be reexpressed as a set of Π_2 sentences in the usual language of Arithmetic.

- So, $Th_{\Pi_2}(\mathbf{I}\Pi_1^-) \vdash \forall a \in \mathcal{I}_1(\mathfrak{A})\ \forall b \in \mathcal{I}_1^1(\mathfrak{A})\ \exists y\, (y = b^a)$.

Characterizing $Th_{\Pi_2}(\mathbf{I}\Pi_1^-)$

Theorem

$$Th_{\Pi_2}(\mathbf{I}\Pi_1^-) \equiv \mathbf{I}\Delta_0 + \forall a \in \mathcal{I}_1(\mathfrak{A})\ \forall b \in \mathcal{I}_1^1(\mathfrak{A})\ \exists y\,(y = b^a).$$

Proof: ($\dashv$):

Case 1: $\mathcal{I}_1^1(\mathfrak{A}) \neq \mathfrak{A}$. It follows from a general property:

$$\left.\begin{array}{c} \mathfrak{A} \models \mathbf{I}\Delta_0 \\ \mathcal{I}_1^1(\mathfrak{A}) \neq \mathfrak{A} \end{array}\right\} \Longrightarrow \mathfrak{A} \models \mathbf{L}\Sigma_1^- \equiv \mathbf{I}\Pi_1^-$$

Case 2: $\mathcal{I}_1^1(\mathfrak{A}) = \mathfrak{A}$. Suppose $I_{\theta(x)} \vdash \varphi$, with $\varphi \in \Pi_2, \theta(x) \in \Pi_1^-$.

We mimic Kaye's proof of Π_2–conservation between $\mathbf{I}\Pi_1^-$ and $\mathbf{I}\Delta_0 + exp$ to construct $\mathfrak{B} \models I_{\theta(x)} + Th_{\Sigma_2}(\mathfrak{A})$. So, $\mathfrak{A} \models \theta$. □

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Proposition

Suppose $T \vdash \mathbf{I}\Delta_0$. The following are equivalent:

1. $\mathfrak{A} \models Th_{\mathcal{B}(\Sigma_1)}(T)$.
2. *There exists $\mathfrak{B} \models T$ such that $\mathfrak{A} \equiv_1 \mathfrak{B}$.*

Proof: $T + Th_{\Sigma_1}(\mathfrak{A}) + Th_{\Pi_1}(\mathfrak{A})$ is consistent. □

Corollary

1. *If $T \vdash \mathbf{I}\Delta_0$ then $Th_{\Sigma_2}(Th_{\Pi_2}(T)) \equiv Th_{\mathcal{B}(\Sigma_1)}(T)$.*
2. $Th_{\Sigma_2}(\mathbf{I}\Delta_0 + exp) \equiv Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Delta_0 + exp) = Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Pi_1^-)$.

Proof: $\mathfrak{A} \equiv_1 \mathfrak{B} \Rightarrow \mathcal{K}_1(\mathfrak{A}) \cong \mathcal{K}_1(\mathfrak{B})$. □

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Proof:

($\vdash$): Pick $c \in \mathcal{K}_1(\mathfrak{A})$ with $a, b \leq c$. Let $\mathfrak{B} \models \mathbf{I}\Pi_1^-$ such that $\mathfrak{A} \equiv_1 \mathfrak{B}$.

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Since $\mathfrak{A} \equiv_1 \mathfrak{B}$, b^a exists in $\mathfrak{A}$.

($\dashv$): $\mathcal{K}_1(\mathfrak{A}) \models exp \vdash Th_{\Pi_2}(\mathbf{I}\Pi_1^-)$. Since $\mathcal{K}_1(\mathfrak{A}) \prec_1 \mathfrak{A}$, $\mathfrak{A} \models Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Pi_1^-)$.

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Separating $Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Pi_1^-)$ and $Th_{\Pi_1}(\mathbf{I}\Pi_1^-)$

Theorem

$Th_{\Pi_1}(\mathbf{I}\Pi_1^-) \nvdash Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Pi_1^-)$.

Proof:

- Given a nonstandard Π_0–minimal element a, we define:

$$f_a(x) = \begin{cases} 2^x & \text{if } x = a \\ 0 & \text{if } x \neq a \end{cases}$$

- $Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Pi_1^-) \vdash \forall x \exists y\, (y = f_a(x))$.

  Since $a \in \mathcal{K}_1(\mathfrak{A})$.

- $Th_{\Pi_1}(\mathbf{I}\Pi_1^-) \nvdash \forall x \exists y\, (y = f_a(x))$.

  Otherwise, $2^a \leq a^k$ by Parikh's Theorem.

Separating $Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Pi_1^-)$ and $Th_{\Pi_1}(\mathbf{I}\Pi_1^-)$

Theorem

$Th_{\Pi_1}(\mathbf{I}\Pi_1^-) \nvdash Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Pi_1^-)$.

Proof:

- Given a nonstandard Π_0–minimal element a, we define:

$$f_a(x) = \begin{cases} 2^x & \text{if } x = a \\ 0 & \text{if } x \neq a \end{cases}$$

- $Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Pi_1^-) \vdash \forall x \exists y\, (y = f_a(x))$.

  Since $a \in \mathcal{K}_1(\mathfrak{A})$.

- $Th_{\Pi_1}(\mathbf{I}\Pi_1^-) \nvdash \forall x \exists y\, (y = f_a(x))$.

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Separating $Th_{\Pi_2}(\mathbf{I}\Pi_1^-)$ and $Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Pi_1^-)$

- Bigorajska's Theorem can be derived from Parikh's Theorem.

$$\mathbf{I}\Delta_0 + \exists u\, \theta(u) \vdash \forall x\, \exists y\, \varphi(x,y)$$
$$\Downarrow$$
$$\mathbf{I}\Delta_0 \vdash \forall x, u\, \exists y\, (\theta(u) \to \varphi(x,y))$$
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Proposition. Suppose $\mathfrak{A} \models T$ and $T \vdash \forall x\, \exists y\, \varphi(x,y)$, with $\varphi \in \Sigma_1$.

1. $T \subseteq \mathcal{B}(\Sigma_1) \;\Rightarrow\; \exists a \in \mathcal{I}_1(\mathfrak{A}),\ \mathfrak{A} \models \forall x > a\, \exists y \leq t(x)\, \varphi(x,y)$.
2. $T = \mathbf{I}\Pi_1^- \;\Rightarrow\; \exists a \in \mathcal{I}_1^1(\mathfrak{A}),\ \mathfrak{A} \models \forall x > a\, \exists y \leq t(x)\, \varphi(x,y)$.

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Separating $Th_{\Pi_2}(\mathbf{I}\Pi_1^-)$ and $Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Pi_1^-)$

Theorem

$Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Pi_1^-) \nvdash Th_{\Pi_2}(\mathbf{I}\Pi_1^-)$.

Proof:

- Given a, c Π_0–minimal and $\delta(z, v) \in \Pi_0$:

$$g(x) = \begin{cases} (x)_3^{(x)_0} & \text{if } \begin{array}{l}(x)_0 = a \wedge (x)_1 = c \wedge (x)_2 \leq (x)_1 \\ (x)_3 = (\mu z)\,(\delta(z, (x)_2))\end{array} \\ 0 & \text{otherwise} \end{cases}$$

- $Th_{\Pi_2}(\mathbf{I}\Pi_1^-) \vdash \forall x\, \exists y\, (y = g(x))$.
  Since $(x)_3 \in \mathcal{I}_1^1(\mathfrak{A})$ and $(x)_0 \in \mathcal{I}_1(\mathfrak{A})$.

- $Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Pi_1^-) \nvdash \forall x\, \exists y\, (y = g(x))$.
  Otherwise, it would follow from Bigorajska's Theorem that $b^a \leq b^k$ for some $b \in \mathcal{I}_1^1(\mathfrak{A}) - \mathcal{I}_1(\mathfrak{A})$.

Separating $Th_{\Pi_2}(\mathbf{I}\Pi_1^-)$ and $Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Pi_1^-)$

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Conclusions

- *Informative* axiomatizations of $Th_{\Pi_2}(\mathbf{I}\Pi_1^-)$ and $Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Pi_1^-)$ in terms of restricted exponentiation.
- Separation theorem: $Th_{\Pi_1}(\mathbf{I}\Pi_1^-) \subset Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Pi_1^-) \subset Th_{\Pi_2}(\mathbf{I}\Pi_1^-)$.
- Descriptions of what total functions of a nonstandard model of $\mathbf{I}\Pi_1^-$ look like.
- Information on the limits of Parikh's and Bigorajska's Thm:
  - $Th_{\mathcal{B}(\Sigma_1)}(\mathbf{I}\Pi_1^-)$ satisfies Bigorajska's Theorem but not Parikh's.
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What about $\mathbf{I}\Pi^-_{n+1}$ for $n > 0$?

- Our results generalize to $n > 0$ for $\mathbf{I}\Sigma_n + \mathbf{I}\Pi^-_{n+1}$

$\mathbf{I}\Delta_0$	$\mathbf{I}\Sigma_n$, $n > 0$
polynomials	functions $y = \mathbb{K}_n(x)$
exponentiation	iterations of $\mathbb{K}_n(x)$

Theorem

1. $Th_{\Pi_{n+2}}(\mathbf{I}\Sigma_n + \mathbf{I}\Pi^-_{n+1}) \equiv \mathbf{I}\Sigma_n + \forall a \in \mathcal{I}_{n+1}\, \forall b \in \mathcal{I}^1_{n+1}\, \exists y\, (y = \mathbb{K}^a_n(b))$.
2. $Th_{\mathcal{B}(\Sigma_{n+1})}(\mathbf{I}\Sigma_n + \mathbf{I}\Pi^-_{n+1}) \equiv \mathbf{I}\Sigma^-_n + \forall a, b \in \mathcal{I}_{n+1}\, \exists y\, (y = \mathbb{K}^a_n(b))$.
3. $Th_{\Pi_{n+1}}(\mathbf{I}\Sigma_n + \mathbf{I}\Pi^-_{n+1}) \subset Th_{\mathcal{B}(\Sigma_{n+1})}(\mathbf{I}\Sigma_n + \mathbf{I}\Pi^-_{n+1}) \subset Th_{\Pi_{n+2}}(\mathbf{I}\Sigma_n + \mathbf{I}\Pi^-_{n+1})$.

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What about $\mathbf{I}\Pi^-_{n+1}$ for $n > 0$?

Theorem (Beklemishev,'99)

$\mathbf{I}\Pi^-_{n+1}$ *is* $\mathcal{B}(\Sigma_{n+1})$*–conservative over* $\mathbf{I}\Sigma^-_n$.

- So, $Th_{\mathcal{B}(\Sigma_{n+1})}(\mathbf{I}\Pi^-_{n+1}) \equiv \mathbf{I}\Sigma^-_n$.
- Questions:
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- A deeper analysis of axiom schemes and inference rules up to definable elements could help.

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