Complexity of real numbers

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Digital vs computational complexity

Digital complexity

Definition

Let Γ_* be a class of relations on N

Let $\alpha > 0$ be a real number

α is $\Gamma.D_b$

when $\lfloor b^n \alpha \rfloor$ mod b are in Γ_*

Digital complexity

Exemple

$$\alpha = \sum_{i=0}^{\infty} \frac{1}{2^{2^i}}$$

is in $FO(+, V_2)^N$

$$\alpha = \sum_{i=0}^{\infty} \frac{1}{2^{i^2}}$$

is not in $FO(+, V_2)^N$

Computational complexity

Definition

Let Γ be a class of functions $N \mapsto N$

Let $\alpha > 0$ be a real number

α is $\Gamma.C$ when

there exist functions f and g in Γ such that

$$\left| \frac{f(n)}{1 + g(n)} - \alpha \right| \leq \frac{1}{1 + n}$$

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Computational complexity

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Exemple

$\sqrt{2}$ in $E^2.C$

E^2 is the second class in Grzegorczyk's hierarchy

Digital complexity

Theorems

$FO(+, V_k)^N.D_b$

• (Shallit, Allouche) are closed under $+$

• (Shallit, Allouche and al.) are not closed under $\times$

• (Adamczevski, Bugeaud 07) contain no algebraic irrational real numbers.

Digital complexity

Theorem

$E^2_*.D_2$ is not closed under $+$

Digital complexity

Conjecture

$Spec.D_2$ contain algebraic real numbers

Elements of $Spec$ are the sets of integers n that are cardinalities of the finite models of some first order formula

Computational complexity

Theorems

- (Ko 83, 89) $P.C$ is a real closed field
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Computational complexity

Theorems

- (Ko 83, 89) $P.C$ is a real closed field
- (Skordev 02) $E^2.C$ is a field
- (Sk 02) Algebraic reals are in $E^2.C$
- (Sk 08) e, γ, π, $\alpha = \sum_{i=0}^{\infty} \frac{1}{2^{i!}}$, ... are in $E^2.C$
- (Sk 08) $E^2.C \subset E^3.C$

Computational complexity

Definition

Let L be the least class of functions on N

- containing constants, proj., succ, $+$
- closed under composition
  and bouded summation

Theorem More , E 98

L is the class of functions on N

- less than a polynomial
- the graph of which is in $(\Delta_0^N)^\sharp$

Computational complexity

Theorems

- (Skordev 08) π is $L.C$

- (Tent, Ziegler 09) $L.C$ is a real closed field